massless particle: only possible to boost to a frame with

$$p^\mu = (E, \underbrace{0, 0}_{\text{little group is } SO(2)}, E) \quad \text{because} \quad p^\mu p_\mu = -\frac{E^2}{c^2} + |p|^2 \overset{1=}{=} -m^2 c^2 = 0$$

only one generator, J_{12}, eigenvalue is called helicity λ. Like spin s, λ is a half integer.

complete state $|p_\mu, \lambda\rangle$

examples: massive:

- $s = 0$ Higgs boson
- $s = 1/2$ electron
- $s = 1$ W and Z bosons

Higgs mechanism

massless:

- $\lambda = 0$ KG field for $m = 0$
- $\lambda = 1$ photon $\rightsquigarrow$ electromagnetic field
- $\lambda = 2$ graviton $\rightsquigarrow$ gravity

Question: What are the Lagrangians for all these theories?

5. Electrodynamics

is governed by Maxwell equations for electric field $\vec{E}$ and magnetic field $\vec{B}$

1) $\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ how to compute ?

$\vec{\nabla} = \vec{e}_1 \partial_1 + \vec{e}_2 \partial_2 + \vec{e}_3 \partial_3$, and same for

$\vec{E} = \vec{e}_1 E_1 + \vec{e}_2 E_2 + \vec{e}_3 E_3$, $\vec{B} = \vec{e}_1 B_1 + \vec{e}_2 B_2 + \vec{e}_3 B_3$

$\vec{e}_i \times \vec{e}_j = \sum_k \varepsilon_{ijk} \vec{e}_k$ $\rightsquigarrow$ for example

$$\partial_1 E_2 - \partial_2 E_1 = -\frac{1}{c} \frac{\partial B_3}{\partial t} \quad \ldots$$

2) $\nabla \cdot \vec{B} = 0$, 3) $\nabla \cdot \vec{E} = \rho$

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0$$

4) $\nabla \times \vec{B} = \frac{1}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$ (all in Heaviside-Lorentz unit system)

1) + 2) homogeneous, solved for $\vec{E} = \vec{B} = 0$

3) + 4) inhomogeneous due to charge ρ and current $\vec{j}$

5.1. Vector and electric potential

2) is solved by $B = \vec{\nabla} \times \vec{A}$

because $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ grad(curl($\vec{A}$)) = 0

1) then implies $\vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$

$$\rightsquigarrow \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \nabla \phi$$

is allowed because
$\nabla \times (\nabla \phi) = 0$
curl (div ϕ) = 0

5.2. Covariant reformulation

$$A^\mu = (\phi, A_1, A_2, A_3), \quad A_\mu = (-\phi, A_1, A_2, A_3)$$

for which we compute $\quad X^\mu = (ct, x^1, x^2, x^3)$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]}$$

$i = 1, 2, 3$

independent components: $F_{0i_{1,2,3}}$ and $F_{ij_{1,2,3}} \; (i<j)$

$$F_{0i} = \frac{\partial A_i}{\partial x^0} - \frac{\partial A_0}{\partial x^i} = \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x^i} = -E_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \sum_k \varepsilon_{ijk} B_k$$

$$\rightsquigarrow F_{12} = B_3, \quad F_{23} = B_1, \quad F_{13} = -B_2$$

Putting all together we get

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

field strength tensor

Question: How do the Maxwell equations look in this formulation?

1) + 2) $\quad \nabla \cdot (\nabla \times \vec{A}) = 0 \quad$ and $\quad \nabla \times (\nabla \cdot \phi) = 0$

are equivalent to

$$\partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) = 0$$

or equivalently $\boxed{3\,\partial_{[\mu} F_{\nu\lambda]} = 0}$ Bianchi identity

3+4) $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$

with $F^{\mu\nu} = -F^{\nu\mu}$, $F^{0i} = -F_{0i}$
$F^{ij} = F_{ij}$

$\boxed{\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu}$ field equation with

$$j^0 = c\,\partial_\nu F^{0\nu} = c(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)$$
$$= c\,\nabla\cdot\vec{E} = c\rho$$

$$\partial_\nu F^{i\nu} = -\frac{1}{c}\frac{\partial}{\partial t}E_i + \sum_{jk}\partial_j B_k \varepsilon_{jki}$$
$$= \left(-\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B}\right)_i = \frac{1}{c} j_i$$

$\leadsto$ $\boxed{j^\mu = (c\rho, j_1, j_2, j_3)}$ current four vector

in vacuum $j^\mu = 0$

Bonus: We can now easily define electro dynamics in dimension other than 4.

5.3. Action

$$\boxed{S = \frac{1}{4}\int d^4x \; F_{\mu\nu} F^{\mu\nu}}$$ vacuum action

check:

$$\delta S = \frac{1}{2}\int d^4x \; \delta F_{\mu\nu} F^{\mu\nu} = \int dx^4 \; \partial_{[\mu} \delta A_{\nu]} F^{\mu\nu}$$

$$= \int dx^4 \; \delta A_\nu \underbrace{\partial_\mu F^{\nu\mu}}_{=0} = 0$$

e.o.m: $\quad \partial_\nu F^{\mu\nu} = 0$