4.4. Representations of the Poincaré group

remember Lorentz transformations:

$$\boxed{\eta_{\rho\lambda} L^{\rho}{}_{\mu} L^{\lambda}{}_{\nu} = \eta_{\mu\nu}}$$

simplest example $L^{\rho}{}_{\mu} = \delta^{\rho}_{\mu}$

now consider $L^{\rho}{}_{\mu} = \delta^{\rho}_{\mu} + \Lambda^{\rho}{}_{\mu}$ $\ll 1$

$$\cancel{\eta_{\mu\nu}} + \eta_{\rho\nu}\Lambda^{\rho}{}_{\mu} + \eta_{\mu\lambda}\Lambda^{\lambda}{}_{\nu} + \mathcal{O}(\Lambda^2) = \cancel{\eta_{\mu\nu}}$$

$$2\Lambda_{(\mu\nu)} = 0$$

$$\leadsto \delta L^{\mu}{}_{\nu} = \frac{i}{2}\Lambda^{\rho\sigma}(J_{\rho\sigma})^{\mu}{}_{\nu} \left(= \Lambda^{\mu}{}_{\nu}\right)$$ with generators

$$(J_{\rho\sigma})^{\mu}{}_{\nu} = i\left(\eta_{\sigma\nu}\delta^{\mu}_{\rho} - \eta_{\rho\nu}\delta^{\mu}_{\sigma}\right)$$ 4x4 matrix

satisfying

$$\boxed{[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}\right)}$$

Lorentz algebra = Lie algebra

3 rotations $(J_{kl})^{\dagger} = J_{kl}$

3 boosts $(J_{0k})^{\dagger} = -J_{0k}$

define: $R^{i} := \varepsilon^{ijk} J_{jk} = \frac{2}{i}(\varepsilon^{i})^{j}{}_{k}$

$$R^1 = \frac{2}{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R^2 = \frac{2}{i}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R^3 = \frac{2}{i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we now find for example: $[R^1, R^2] = 2i R^3$

$$\boxed{[R^m, R^n] = 2i \varepsilon^{mn}{}_{l} R^l}$$ Lie algebra $SO(3)$

remember angular momentum in QM:

$$[L_l, L_m] = \hbar i \sum_{n=1}^{3} \varepsilon_{lmn} L_n$$

there we also know $L^2 = L_1^2 + L_2^2 + L_3^2$

quadratic Casimir invariant

with the property $[L^2, L_i] = 0$

commuting sets of hermitian operators can be diagonalized simultaneously.

i.e. L^2 and L_z with eigenvalues

$\hbar m_j$, $\quad m_j = -j, -j+1, \dots, j$ $\quad$ $\boxed{\text{spin}}$

$j(j+1)\hbar^2$ for $j = 0, 1/2, 1, 3/2, \dots$

Same holds for our R^i's with

$$(R^1)^2 + (R^2)^2 + (R^3)^2 = 8 \cdot \mathbb{1} = 1 \cdot (1+1) \cdot 2^2$$

$\leadsto j = 1$ and eigenvalues of R^3 from

$$\det(R^3 - \lambda \mathbb{1}) = -\lambda(\lambda^2 - 4)$$

$\lambda_j = 2 \cdot m_j$ with $m_j = -1, 0, 1$

this is not the smallest representation, we can also have

$$[\sigma^i, \sigma^j] = 2i\, \varepsilon^{ij}{}_k\, \sigma^k$$ ~ Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 = 3 \cdot \mathbb{1}$$
$$= 1/2\,(1/2 + 1) \cdot 4 \cdot \mathbb{1}$$

Observation: 1) one Lie algebra (here SO(3)) but many representations

2) distinguish between them with Casimir invariants

Question: How to find new representations?

$\rightsquigarrow$ tensor product

$$R'^i = R^i \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes R^i \; ; \; R^i = \begin{pmatrix} R^i_{11} & \cdots & R^i_{13} \\ \vdots & \ddots & \vdots \\ R^i_{31} & \cdots & R^i_{33} \end{pmatrix}$$

$$\begin{pmatrix} R^i_{11}\,\mathbb{1}_3 & \cdots & \boxed{R^i_{13}\,\mathbb{1}_3} \\ \vdots & \ddots & \vdots \\ R^i_{31}\,\mathbb{1}_3 & \cdots & R^i_{33}\,\mathbb{1}_3 \end{pmatrix}$$ 3x3 block

$\rightsquigarrow$ in total 9x9

$$[\overbrace{R^i \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes R^i}^{R'^i}, \overbrace{R^j \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes R^j}^{R'^j}] =$$
$$([R^i, R^j]) \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes ([R^i, R^j])$$
$$= 2i\, \varepsilon^{ij}{}_k\, R'^k$$

again eigenvalues of Casimir

$\rightsquigarrow$ 5 times $24 = 2 \cdot (2+1) \cdot 4$ $\qquad 1 \times 0 = 0 \cdot (0+1) \cdot 4$

3 times $8 = 1 \cdot (1+1) \cdot 4$

$$\overbrace{3 \otimes 3}^{\text{representation}} = 1 \oplus 3 \oplus 5$$

irreducible representation ~ fundamental building blocks

dimensions; 1: spin 0; 3: spin 1; 5: spin 2

Question: What happens for the full Poincaré group?

1) additional generators P_μ

with commutator $[P_\mu, P_\nu] = 0$ and

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu)$$

2) unitary representations can be found by (important for quantization)

2a) choose a frame in which all non-compact (boosts and translations) transformations are fixed

2b) Only the remaining compact transformations (little group, in 4-dim. $O(3)$ / $O(2)$) need to be considered.

massive and massless particle

two options:

massive particle: boost to 4-momentum

$$p^\mu = (\underbrace{m}_{\text{rest mass}}, \boxed{0, 0, 0})$$

this form is only preserved by rotations $\in O(3)$

representation fixed by spin j and
mass m

example KG field, spin 0 and mass m