I.) find a hyper plane with no roots on it
$\rightsquigarrow$ splits root system into two halfs

$V_{\pm}$ ↙ $\underline{\text{positive roots}}$ $\rightsquigarrow$ $E_\alpha \hat{=}$ $\underline{\text{raising operator}}$
↖ $\underline{\text{negative roots}}$. $E_{-\alpha} \hat{=}$ $\underline{\text{lowering operator}}$

II.) simple root = all roots that cannot be written as sum of two or more positive roots

Independent of the choice of hypersurface there are rank $\mathfrak{g}$ simple roots. They form a basis of Φ.

The $\underline{\text{Cartan matrix}}$ encodes the non-orthogonality of the simple roots
$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad i,j = 1, \dots, \text{rank}\,\mathfrak{g}$$

4.7. su(N) representations

The $\underline{\text{Lie group } SU(N)}$ is defined by its action on a N-component vector v with complex components v^i:
$$M: v \mapsto v' = M \cdot v \quad , \quad v^i \mapsto v'^i = M^i{}_j v^j ,$$
with $M^\dagger M = \mathbb{1}$ and $\det M = 1$.

$\rightsquigarrow$ For the $\underline{\text{Lie algebra } su(N)}$ we have: $X \in su(N)$
$$X: \ v \mapsto v' = \rho(x)\, v = X\, v$$
$$v^i \mapsto v'^i = X^i{}_j v^j ,$$
with $X^\dagger + X = 0$ and $\operatorname{tr} X = 0$

remarks:
- fundamental representation of su(N)
- $V = \mathbb{C}^N$ is the fundamental module
- also called $\underline{\text{vector representation}}$

remember 1.2.2.

We usually distinguish $\underset{\sim}{\text{irreps}}$ by their dimension. If there are several with the same dim., we add decoration like $\bar{\ }$, $'$ ...

$$\overset{"}{(N)}$$

All other irreps can be derived from fundamental representation

A) I.e. take $v \otimes w \in (N) \otimes (N)$

and decompose into (anti-) symmetric part ↗ 3.1. gl(N) reps.

$$(v \vee w)^{ij} = \frac{1}{2}\left(v^i \otimes w^j + v^j \otimes w^i\right) \quad \text{symmetric}$$

$$(v \wedge w)^{ij} = \frac{1}{2}\left(v^i \otimes w^j - v^j \otimes w^i\right) \quad \text{anti-symmetric}$$

both define irreps and we write:

$$(N) \otimes (N) = \underbrace{\left(\tfrac{1}{2} N(N+1)\right)}_{\text{sym.}} \oplus \underbrace{\left(\tfrac{1}{2} N(N-1)\right)}_{\text{anti-sym.}}$$

B) conjugation

i.e. of the fundamental $(N) \to (\bar{N})$

(N): $v \mapsto X \cdot v$

$(\bar{N})$: $\bar{v} \mapsto \bar{v}(-X)$, $\quad \bar{v}_i \mapsto \bar{v}_j (-X)^j{}_i$

last lecture we had: $\quad w \mapsto (-X^T) w$

$$\rightsquigarrow \quad \bar{v} := w^T \; //$$

A + B) $\quad (\bar{N}) \otimes (\bar{N}) = \left(\overline{\tfrac{1}{2} N(N+1)}\right) \oplus \left(\overline{\tfrac{1}{2} N(N-1)}\right)$

or $(N) \otimes (\bar{N})$: $\quad t^i{}_j \mapsto \sum_k X^i{}_k t^k{}_j + \sum_e (-X)^e{}_j t^i{}_e$

where we decompose: $\quad t^i{}_j = \frac{1}{N} s \, \delta^i{}_j + ad^i{}_j$

with

$$s := \operatorname{tr}(t) = \sum_k t^k{}_k \quad \text{and} \quad ad^i{}_j := t^i{}_j - \frac{1}{N} \operatorname{tr}(t) \, \delta^i{}_j$$

- s is called singlet because the module is one (= single) dimensional = trivial representation.

$$s = \sum_i t^i{}_i \mapsto s' = \sum_j \left(\sum_K X^j{}_K t^K{}_j + \sum_e (-X)^e{}_j t^j{}_e \right)$$
$$= \sum_{j,K} \left(X^j{}_K t^K{}_j - X^j{}_K t^K{}_j \right) = 0$$

- $ad^i{}_j$ is the adjoint module (please check) with $\dim(ad) = N^2 - 1$

Note: $\delta^i{}_j$ is an invariant tensor of $su(N)$.
It does not transform. ⇝ used for singlet

The second invariant tensor of $su(N)$ is called Levi-Cevita tensor $\varepsilon_{i_1 \dots i_N}$

It gives rise to the singlet $(N)^{\wedge N} = \underbrace{(N) \wedge \dots \wedge (N)}_{N\text{-times}} = (1)$

$$\hat{=} \quad V^{i_1} \wedge \dots \wedge V^{i_N} = s\, \varepsilon^{i_1 \dots i_N} .$$

$\to$ $\boxed{su(N) \text{ } N\text{-times totally anti-symm. irrep} = \text{trivial irrep}}$

In a similar spirit we find:

$$(N)^{\wedge(N-1)} = \overbrace{(N) \wedge \dots \wedge (N)}^{N-1} = (\bar{N})$$
$$V^{i_1} \wedge \dots \wedge V^{i_{N-1}} = \varepsilon^{i_1 \dots i_{N-1} j}\, \bar{V}_j$$
$$\bar{V}_j = \frac{1}{(N-1)!} \varepsilon_{i_1 \dots i_{N-1} j} V^{i_1} \dots V^{i_{N-1}}$$

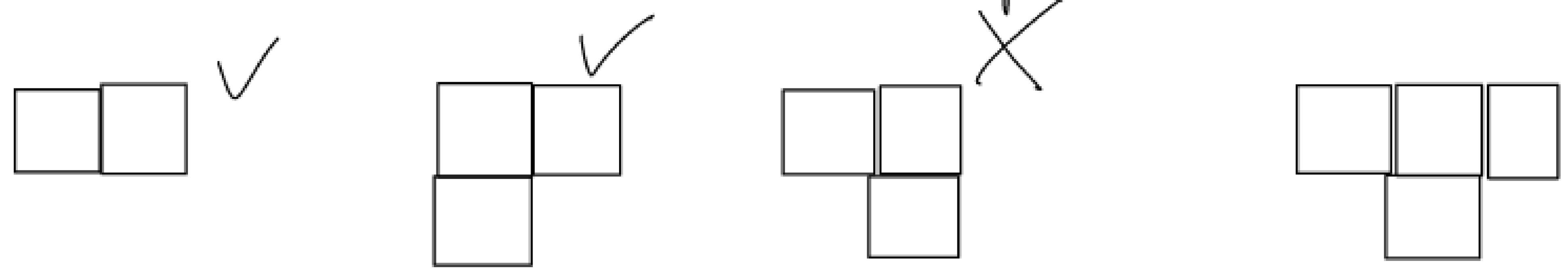
4.8. Young tableaux

We have seen: (anti-) symmetrising in tensor products gives rise to irreps.

Idea: diagrammatic representation

Rules: • A rank n tensor $t^{i_1 \dots i_n}$ is represented by n boxes.

- Draw them in columns such that their number does never increase from left to right.



- anti-symmetrise over columns & symmetrise over rows:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \mathrel{\hat{=}} \big(() - (13) \big)\big(() + (12) \big) = () + (12) - (13) - (132)$$

$$t'^{\,i_1 i_2 i_3} = t^{i_1 i_2 i_3} + t^{i_2 i_1 i_3} - t^{i_3 i_2 i_1} - t^{i_3 i_1 i_2}$$

with $t'^{\,i_1 i_2 i_3} = t'^{\,i_2 i_1 i_3} = -t'^{\,i_3 i_2 i_1}$

$$\boxed{\text{For } su(N) \text{ at most } N \text{ boxes in column!}}$$

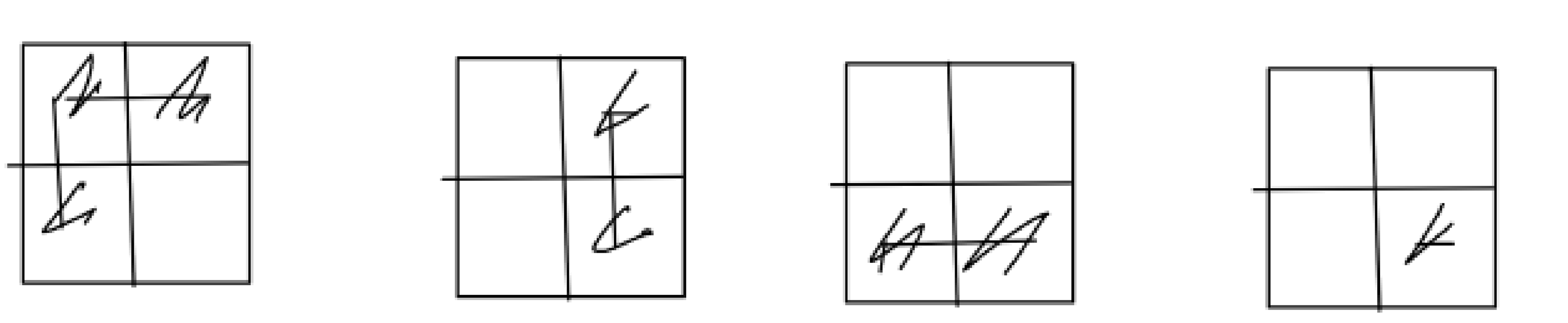
$$\square = (N), \quad N\left\{\begin{array}{c}\square\\ \vdots \\ \square\end{array}\right. = (1), \quad N-1\left\{\begin{array}{c}\square\\ \vdots \\ \square\end{array}\right. = (\bar{N})$$

irrep ~ different Young tableaux

- dim. of corresponding irrep:

$$\dim = \frac{\text{numerator}}{\text{denominator}}$$

multiply hook length for each cell:



$$3 \cdot 2 \cdot 2 \cdot 1 = 12$$

- start with N in the top left corner.
- Add one when you go $\longrightarrow$ substract one when you go $\downarrow$

N	$N+1$
$N-1$	N

- multiply $= N^2(N^2-1)$

$$\dim(\boxplus) = \frac{1}{12} N^2 (N^2 - 1)$$