6. Conformal Symmetry

Review: GR $\boxed{\text{space-time}}$ ← Poincaré (translations + Lorentz rotations)

$\times$

YM $\boxed{\text{internal}}$ ← Gauge group

Coleman-Mandula: Properties of S-matrix and Lie algebras
(no-go) ⇝ only direct product between Poincaré and internal symmetry.

Evaded by

conformal symmetry (no S-matrix) today

super symmetry (super Lie algebra) next lecture

Haag-Łopuszański-Sohnius

6.1. Conformal algebra

transformations that leave $g_{\mu\nu}$ inv. up to rescaling

$$g_{\mu\nu}(x) \to e^{2\sigma(x)} g_{\mu\nu}(x)$$

preserves angles and thus the causal structure

in flat space ($g_{\mu\nu} = \eta_{\mu\nu}$) $\quad \delta_\varepsilon g_{\mu\nu} = L_\varepsilon g_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$

$$\rightsquigarrow \quad \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma(x)\, \eta_{\mu\nu} \qquad \text{for generator } \varepsilon_\mu(x)$$

for $d > 2$
$$\varepsilon^\mu = \underbrace{a^\mu}_{P^\mu} + \underbrace{\omega^\mu{}_\nu x^\nu}_{J_{\mu\nu}} + \underbrace{\lambda x^\mu}_{D \text{ (dilation)}} + \underbrace{b^\mu x^2 - 2(b\cdot x) x^\mu}_{K^\mu \text{ (special conformal)}}$$

$b_\mu x^\mu$

In addition to the Poincaré algebra we get:

$$[J_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu)$$
$$[D, P_\mu] = i P_\mu, \quad [D, K_\mu] = -i K_\mu, \quad [D, J_{\mu\nu}] = 0$$
$$[K_\mu, K_\rho] = 0, \quad [K_\mu, P_\nu] = -2i(\eta_{\mu\nu} D - J_{\mu\nu})$$

generates $so(d,2)$ with generators $J_{AB} = -J_{BA}$

$$\eta_{AB} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{\mu}, -1)$$

$J_{\mu\nu} \sim$ Lorentz $so(d-1,1)$

$J_{d(d+1)} = -D$, $J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu)$, and

$J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu)$

Finite transformations: $x^\mu \to \lambda x^\mu$ (scaling)

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b\cdot x + b^2 x^2}$$

can become 0 !

and inversion ($\mathbb{Z}_2$) $\quad x^\mu \to \frac{x^\mu}{x^2} \quad$ ($so(d,2) \rightsquigarrow O(d,2)$)

$\rightsquigarrow$ conformal compactification, add "infinity"

$\mathbb{R}^d$ one point $\quad 0 = x^\mu x^\nu \delta_{\mu\nu}$ (group $O(d+1,1)$)

$\mathbb{R}^{d-1,d}$ light cone $\quad 0 = x^\mu x^\nu \eta_{\mu\nu}$

for $d=2$:
(Eucledian signature) $\quad \underbrace{\partial_0 \varepsilon_1 = -\partial_1 \varepsilon_0 \quad \text{and} \quad \partial_0 \varepsilon_0 = \partial_1 \varepsilon_1}_{\text{Cauchy-Riemann diff. eqs.}}$ (1)

$\varepsilon = \varepsilon^0 + i\varepsilon^1$, $\quad z = x^0 + ix^1$, $\quad \bar{z} = x^0 - ix^1$

(1) $\rightsquigarrow$ $\underbrace{\frac{\partial}{\partial \bar{z}}}_{\partial_{\bar{z}}} \varepsilon(z,\bar{z}) = 0 \qquad \varepsilon(z)$ is <u>holomorphic</u>

Laurent series: $\varepsilon(z) = -\sum_{n\in\mathbb{Z}} \underbrace{\varepsilon_n z^{n+1}}_{l_n}$, $\quad \bar{\varepsilon}(\bar{z}) = -\sum_{n\in\mathbb{Z}} \underbrace{\bar{\varepsilon}_n \bar{z}^{n+1}}_{\bar{l}_n}$

$[l_n, l_m] = (m-n)\, l_{m+n}$, same for $\bar{l}_n$ while

$[l_n, \bar{l}_m] = 0$

Witt algebra with $\{l_{-1}, l_0, l_1\} = sl(2,\mathbb{R})$ subalgebra

after quantization $[L_n, L_m] = (m-n)L_{m+n} + \underbrace{\frac{c}{12}(m^3-m)\delta_{m+n,0}}_{\text{central extension}}$

with central charge c

6.2. Transformations of the field

we impose: $[D, \phi(0)] = -i\underbrace{\Delta}_{\text{scaling dimension}}\phi(0)$

or equally $x' = \lambda x, \quad \phi'(x') = \lambda^{-\Delta}\phi(x)$

$[K_\mu, \phi(0)] = 0$ conformal primary only

Why? Unitarity representation has bound $\boxed{\Delta \geq \frac{d-2}{2}}$ (i.e. saturated for free scalar)

remember $[D, P_\mu] = \underbrace{(+1)}_{\text{increases } \Delta} i P_\mu$, $[D, K_\mu] = \underbrace{(-1)}_{\text{decreases } \Delta} i K_\mu$

$\phi(0)$ is the lowest weight state in an irreducible representation. By applying P_μ (derivative) on it we get all the other states in the irrep ($\hat{=}$ conformal decendents).

What about $\phi(x)$? $\phi(x) = T(x)^{-1}\phi T(x)$

$T = \exp(-iP_\mu x^\mu)$

↳ i.e. $[D, \phi(x)] = -i\Delta\phi(x) - ix^\mu\partial_\mu\phi(x)$ and more in the tutorial.

6.3. Correlation functions

Conformal symmetry severely restricts ♡

$$\delta \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \sum_{i=1}^{n} \langle \phi_1(x) \dots \underbrace{\delta\phi_i(x_i)}_{\text{conformal transformation}} \dots \phi_n(x_n) \rangle = 0$$

(Ward identity)

for finite dilations, we then find for $n=2$

$$\langle \phi(x_1)\, \phi(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi(\lambda x_1)\, \phi(\lambda x_2) \rangle$$

moreover ↑ can only depend on $(x_1 - x_2)^2$

$$\rightsquigarrow \boxed{\langle \phi(x_1)\, \phi(x_2) \rangle = \frac{C_{\phi_1\phi_2}}{(x_1-x_2)^{\Delta_1+\Delta_2}}} = \left[(x_1-x_2)\cdot(x_1-x_2)\right]^{\frac{\Delta_1+\Delta_2}{2}}$$

constants

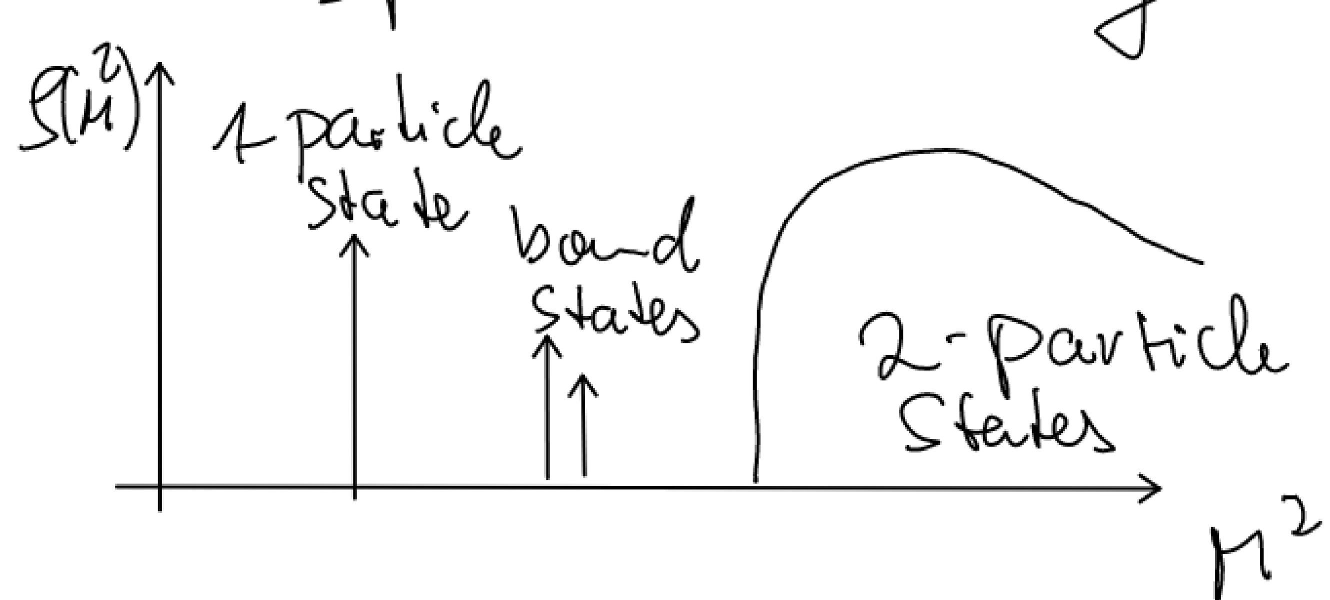
$C_{\phi_1\phi_2} = C_{\phi_2\phi_1}$ $\rightsquigarrow$ diagonalize ! and rescale

$$\rightsquigarrow \boxed{\langle \mathcal{O}(x_1)\, \bar{\mathcal{O}}(x_2) \rangle = \frac{1}{(x_1-x_2)^{2\Delta}}}$$

; compare with just Poincaré

$$\langle \phi(x_1)\, \phi(x_2) \rangle = \int \frac{dM}{2\pi}\, \rho(M^2)\, D_F(x_1-x_2, M^2)$$

spectral density



Similar for 3-point functions with

$$\boxed{\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{(x_1-x_2)^{\Delta_1+\Delta_2-\Delta_3}(x_2-x_3)^{-\Delta_1+\Delta_2+\Delta_3}(x_1-x_3)^{\Delta_1-\Delta_2+\Delta_3}}}$$