Now we look at the complex scalar field with

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \bar{\phi} - m^2 \phi \bar{\phi}$$

and $\quad \phi(x) \to \phi'(x) = \phi(x) + i\alpha(x)\,\phi(x)$

Again the measure does not change, $\mathcal{D}\phi = \mathcal{D}\phi'$

repeating the steps above, we obtain

$$\langle \partial_\mu j^\mu(x)\, \phi_1\, \bar{\phi}_2 \rangle = (-i) \langle i\phi_1\, \delta(x-x_1)\, \bar{\phi}_2 - i\phi_1\, \bar{\phi}_2\, \delta(x-x_2) \rangle$$

and $j^\mu = i\left(\phi \partial^\mu \bar{\phi} - \bar{\phi}\partial^\mu \phi\right)$

for a general Lagrangian and symmetry

$$\boxed{\langle \partial_\mu j^\mu\, \phi_1\, \phi_2 \rangle = (-i) \langle \delta\phi_1\, \delta(x-x_1)\, \phi_2 + \phi_1\, \delta\phi_2\, \delta(x-x_2) \rangle}$$

contact term

5.8. The Electromagnetic Field

We encountered the em field in lecture 2 but did not compute its propagator yet. $\to$ today with path integral

↯ challenge: gauge symmetry requires care in identifying propagation (physical) degrees of freedom

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\, A_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)$$

Fourier transformation

$S = 0 \quad$ for $\quad \tilde{A}_\mu(k) = k_\mu \alpha(k) \quad \rightsquigarrow$ path integral

$Z_0 = \int \mathcal{D}A\, e^{iS[A]}$ diverges

or equally: $\quad \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right)\tilde{D}_F^{\nu\rho} = i\delta_\mu{}^\rho$

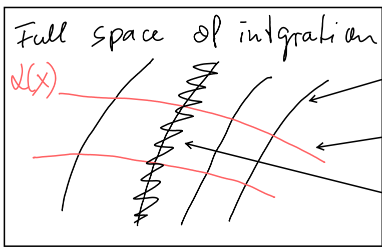
see last lecture

cannot be solved. Reason gauge symmetry (remember)

gauge transformation: $A_\mu(x) \to A_\mu(x) + \underbrace{\frac{1}{e}\partial_\mu \alpha(x)}$

$\sim \partial_\mu \alpha(x)$ or $k_\mu \alpha(k)$ after FT

Idea of Faddeev & Popov:



physical part

gauge orbit

gauge fix slice; fixed by $G(A) = 0$

Idea: insert δ-function into the path integral which restricts to gauge fixed slice

1) Define path integral version of δ-function

$$1 = \int \mathcal{D}\alpha(x)\, \delta(G(A^\alpha))\, \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

$$A^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

EX

originates from $1 = \left(\prod_i \int da_i\right) \delta^{(n)}(\vec{g}(\vec{a}))\, \det\left(\frac{\partial g_i}{\partial a_j}\right)$

$$Z_0 = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \underbrace{\int \mathcal{D}\alpha}_{\text{divergent}} \underbrace{\int \mathcal{D}A\, e^{iS[A]}\, \delta(G(A))}_{\text{physical part}}$$

with $G(A) = \partial^\mu A_\mu(x) - \omega(x)$ ← any scalar function

$$Z_0 = \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

2) Average over gauge choices $\omega(x)$

$$Z_0 = \overbrace{N(\xi)\int \mathcal{D}\omega \, \exp\left[-i\int d^4x \frac{\omega^2}{2\xi}\right]}^{1} \cdot Z_0$$

Gauss function centered at $\omega(x)=0$

$$= N(\xi) \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A \, e^{iS[A]} \exp\left[-i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]$$

normalisation factor

Correlation functions:

$$\langle 0|T\, \mathcal{O}(A)\,|0\rangle = \frac{\int \mathcal{D}A \, \mathcal{O}(A)\, e^{iS}}{Z_0}$$

gauge invariant!

$$= \frac{\int \mathcal{D}A \, \mathcal{O}(A)\exp\Big[\overbrace{i\left(S - \frac{1}{2\xi}\int d^4x\,(\partial^\mu A_\mu)^2\right)}^{i \cdot S_{eff}}\Big]}{\int \mathcal{D}A \;\;\text{---}\;''\;\text{---}}$$

All inconvinient factors cancel!

for S_{eff} $\quad \left[-k^2 g_{\mu\nu} + (1-\xi^{-1})k_\mu k_\nu\right]\tilde{D}_F^{\nu\rho}(k) = i\,\delta_\mu{}^\rho$

which is solved by

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$

in particular $\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge ← we use this one

5.9. Non-abelian Gauge Fields

similar derivation with

- $(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f_{bc}{}^a A^b_\mu \alpha^c = A^a_\mu + \frac{1}{g}D_\mu \alpha^a$
- $G(A) = \partial^\mu A^a_\mu(x) - \omega^a(x)$

propagator: $\tilde{D}_F^{\mu\nu;ab}(k) = \tilde{D}_F^{\mu\nu}(k)\,\delta^{ab}$

$\xi = 1$ is now called Feynman-'t Hooft gauge

↯ : $\dfrac{\delta G(A^\alpha)}{\delta \alpha} = \dfrac{1}{g}\partial^\mu D_\mu$ now depends on A_μ

⇝ $\det\left(\dfrac{\delta G(A^\alpha)}{\delta\alpha}\right)$ cannot be factored out, instead

$$= \int \mathcal{D}c\, \mathcal{D}\bar{c}\, \exp\left[i\int d^4x\, \bar{c}\, \partial^\mu D_\mu c\right]$$

have to be Grassmann numbers

on the other hand c behaves like a complex scalar field (spin = 0) → $\boxed{c \text{ violates spin-statistic theorem}}$

Faddeev-Popov ghosts

$$\boxed{\mathcal{L}_{eff} = -\tfrac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 - \bar{c}^a \partial^\mu D_\mu c^a}$$

ghost = wrong sign for kinetic term

Feynman rules in Exercise!