4.5 Root System

To work with Lie algebras, we use an eigenbasis $\{\tilde{T}_a\}$ with

$$\mathrm{ad}_X(\tilde{T}_a) = [X, \tilde{T}_a] = \xi^a \tilde{T}_a$$

eigenvalues for ξ^a; $\tilde{T}_a$ ← eigenvectors

⇝ solve characteristic equation $\det(\mathcal{S}_X - \mathbb{1}\vec{\xi}) = 0$

[remember matrix rep. $\mathcal{S}_a{}^b T_b = \mathrm{ad}_X(T_a)$]

Solutions should always exist ⇝ $\mathbb{C}$ (smallest algebraic complete field) instead of $\mathbb{R}$.

Remember from quantum mechanics

Maximal set of generators X that solves the eigenvalue equation has to be linearly independent & commute

They span the Abelian Cartan subalgebra with $[H_i, H_j] = 0$. Its dimension is called rank.

$i, j = 1, \ldots, r = \mathrm{rank}\, G = \dim G_0$

We now can write

$$[H, Y] = \mathrm{ad}_H(Y) = \alpha_Y(H)\, Y$$

eigenvalue & —"— vector

eigenvalues are $\boxed{\text{roots}}$ of the characteristic equation

$$\alpha_Y : \mathfrak{g}_0 \to \mathbb{C} \quad \text{or} \quad \alpha_Y \in \mathfrak{g}_0^*$$

Idea: decompose Lie algebra $\mathfrak{g}$ into

$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} \mid [H, Y] = \alpha(H) Y, \forall h \in \mathfrak{g}_0 \}$

root space decomposition of g

for simple Lie algebras it has the properties

1) $\mathrm{span}_{\mathbb{C}}(\Phi) = \mathfrak{g}_0^*$ (usually more roots than rank $\rightsquigarrow$ no basis)

2) For any $\alpha \in \Phi$, there is a $Y_{-\alpha} \in \mathfrak{g}$ such that $\overbrace{K(Y_\alpha, Y_{-\alpha})}^{}\neq 0$. Killing metric

3) The only multiples of $\alpha \in \Phi$ which are roots are $\pm\alpha$.

4) The root spaces $\mathfrak{g}_\alpha$ are one dimensional.

4.6. Cartan-Weyl basis

I. basis for $\mathfrak{g}_0$, with basis elements H_α

such that $\alpha(H) = C_\alpha K(H_\alpha, H) \quad \forall H \in \mathfrak{g}_0$

normalization constant, fixed later

$\rightsquigarrow$ non-degenerate pairing on $\mathfrak{g}_0^*$

$$(\alpha, \beta) := C_\alpha C_\beta K(H_\alpha, H_\beta) = C_\alpha \beta(H_\alpha) = C_\beta \alpha(H_\beta)$$

II. for each $H_\alpha \in \mathfrak{g}_0$ there is a root E_α with

$$[H, E_\alpha] = \alpha(H) E_\alpha$$

3) $\Rightarrow E_{-\alpha}$ is also a root $\{E_\alpha, E_{-\alpha}, H\}$ generate

$sl(2, \mathbb{C})$ subalgebra.

check:

$$K(H,[E_\alpha,E_{-\alpha}]) = K([H,E_\alpha],E_{-\alpha})$$
ad-inv. of $K(\cdot,\cdot)$ $$= \alpha(H)\, K(E_\alpha,E_{-\alpha}) \neq 0$$ because of 2)

$\rightsquigarrow$ $[E_\alpha,E_{-\alpha}] = C_\alpha\, K(E_\alpha,E_{-\alpha})\, H_\alpha$ because Killing form is non-degenerate (simple Lie algebra)

$$[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha)\, E_{\pm\alpha} = \pm \underbrace{\frac{(\alpha,\alpha)}{C_\alpha}}_{2} E_{\pm\alpha}$$

Standard normalization for $sl(2,\mathbb{C})$

$C_\alpha = 1/2\,(\alpha,\alpha)$ and therefore $$\boxed{\alpha(H_\beta) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}}$$

What about $[E_\alpha,E_\beta]$? Here we use another basis for $\mathfrak{g}_0 = \mathrm{span}_{\mathbb{C}}\{H_i\}$ with

$$[H_i,E_\alpha] = \alpha_i E_\alpha \quad \text{and} \quad \alpha_i := \alpha(H_i)$$

$$[H_i,[E_\alpha,E_\beta]] = -[E_\alpha,[E_\beta,H_i]] - [E_\beta,[H_i,E_\alpha]]$$
Jacobi identity
$$= (\alpha_i+\beta_i)[E_\alpha,E_\beta]$$

For $\alpha+\beta \neq 0$, $[E_\alpha,E_\beta]$ is proportional to the generator $E_{\alpha+\beta}$ of the rootspace $\mathfrak{g}_{\alpha+\beta}$, provided $\alpha+\beta\in\Phi$. To summarize:

$$\boxed{\begin{aligned} &[H_i,E_\alpha] = \alpha_i E_\alpha, \qquad [H_i,H_j] = 0 \\ &[E_\alpha,E_{-\alpha}] = H_\alpha = \underbrace{\alpha^{\vee i}}_{\text{co-root}} H_i \quad [E_\alpha,E_\beta] = \begin{cases} e_{\alpha,\beta}\, E_{\alpha+\beta}, & \alpha+\beta\in\Phi \\ 0, & \alpha+\beta\notin\Phi \end{cases}\end{aligned}}$$

Cartan-Weyl basis with normalization $K(E_\alpha,E_{-\alpha}) = 1$

Example $sl(3,\mathbb{C})$ ~ complexification of $su(3)$
~ gauge group of QCD.

generators are traceless, real 3×3 matrices; in total

$$\underbrace{2 \text{ diagonal}}_{\text{Cartan sub.}} + \underbrace{3 \text{ upper triangular} + 3 \text{ lower tri}}_{\text{non-zero roots}} = 8$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Killing form: $K(X,Y) = \mathrm{Tr}(X\cdot Y)$

we take $C_\alpha = 1$ and therefore $(H_i, H_j) = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} \end{pmatrix}$

taking $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

we find:
$$[H_1, E_1] = \boxed{2}\, E_1 \qquad [H_1, E_2] = \boxed{-1}\, E_2$$
$$[H_2, E_1] = \boxed{-1}\, E_1 \qquad [H_2, E_2] = \boxed{2}\, E_2$$

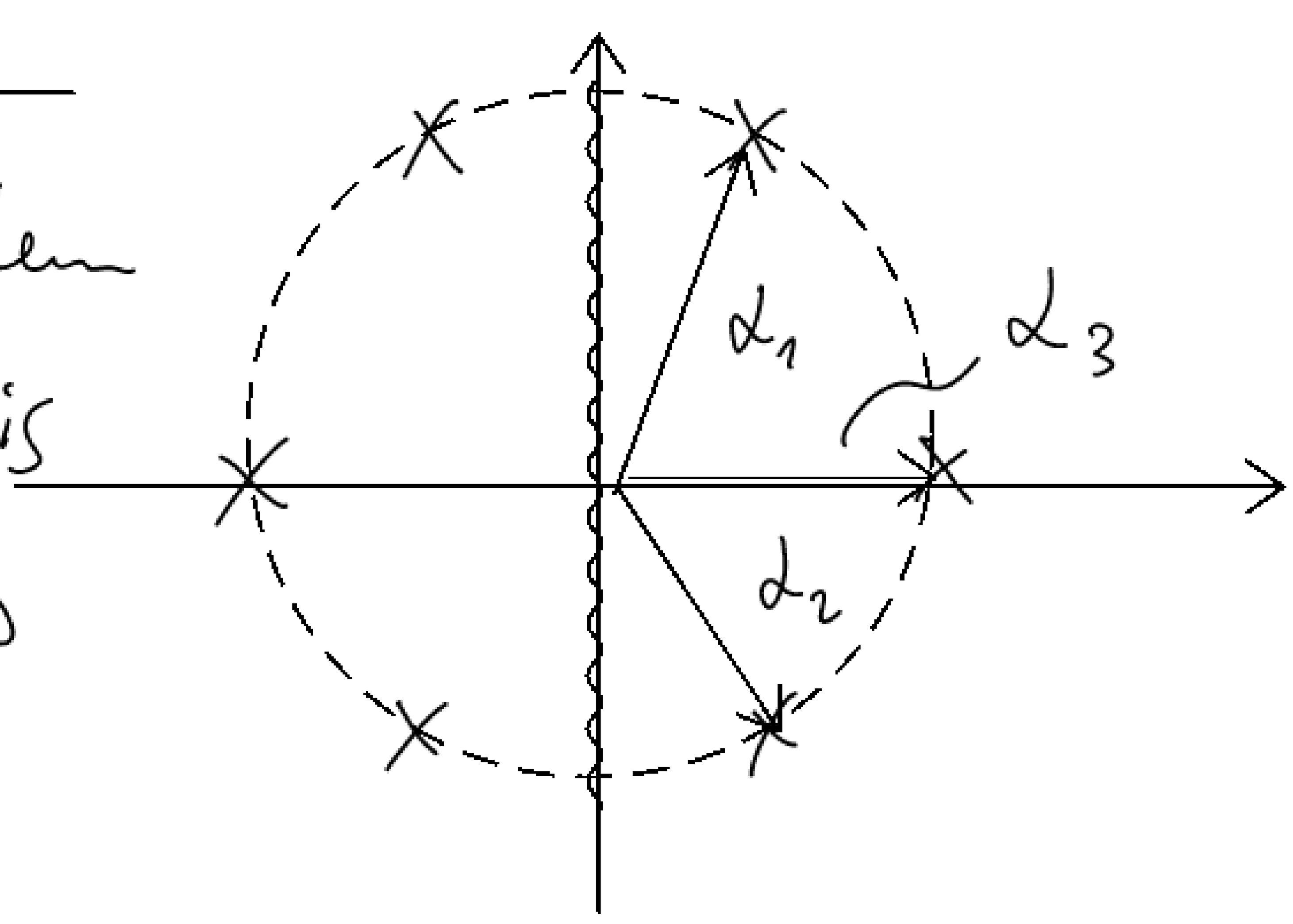
$E_{-1} = E_1^T$, $E_{-2} = E_2^T$ check $K(E_\alpha, E_{-\beta}) = \delta_{\alpha\beta}$

finally $E_3 = [E_1, E_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{-3} = E_3^T$

Homework: check all the other relations

4.6. Simple roots

$sl(2,\mathbb{C})$ root system in an adapted basis for Cartan generators



Question: What is the minimal set of roots to find all others $\rightsquigarrow$ simple roots