5.6. Quantisation of Fermions

5.6.1. Grassmann numbers

remember 2.4.: for Dirac field $\{\hat{a}^r_{\vec{p}}, \hat{a}^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta(\vec{p}-\vec{q})\delta^{rs}$
anti-commutator

Fine for operators, but in path integral we have just ordinary numbers (like $\hat{\phi}$ vs. ϕ from last two lectures).

Question: What shall we do there to implement fermionic fields?

Answer: Grassmann numbers! $\boxed{\theta\eta = -\eta\theta}$

Implies that $\theta^2 = 0$ and that any function $f(\theta)$ can be written as

$$f(\theta) = A + \theta B \quad \text{(Taylor expansion)}$$

We further need: 1) Integration $\int d\theta\, f(\theta)$

$$= \int d\theta (A + B\theta) \text{ should be invariant under } \theta \to \theta + \eta$$

$$= \int d\theta ([A + B\eta] + B\theta) \stackrel{!}{=} B$$

$$\boxed{\int d\theta\, (A + B\theta) = B}$$

multiple integrals: $\int d\theta \int d\eta\, \eta\theta = 1$

2) Differentiation $\frac{\partial}{\partial\theta} f(\theta) = \frac{\partial}{\partial\theta}(A + B\theta) = B$

for Grassmann numbers integration = differentiation

3) complex Grassmann numbers

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \quad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}$$

4) complex conjugation $(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$

we can now evaluate the Grassmann version of a Gaussian integral

$$I = \int d\theta^* d\theta \, e^{-\theta^* b \theta} \xrightarrow{\text{Taylor expand}} = \int d\theta^* d\theta \left(1 - \theta^* b \theta\right)$$

$$= \int d\theta^* d\theta \left(1 + \theta b \theta^*\right) = b$$

remember: if θ would be $\in \mathbb{C}$, $I = \frac{2\pi}{b}$

$$\int d\theta^* d\theta \; \theta\theta^* e^{-\theta^* b \theta} = 1 = \frac{1}{b} \cdot b$$

now in N dimensions $I = \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j}$

I) diagonalise B_{ij} with unitary transformation

$$\theta_i' = U_{ij}\theta_j$$

$$\prod_i \theta_i' = \frac{1}{N!}\varepsilon^{ij\dots l}\theta_i'\theta_j'\cdots\theta_l' = \dots = (\det U)\left(\prod_i \theta_i\right)$$

$$\rightsquigarrow I = \left(\prod_i \int d\theta_i'^* d\theta_i'\right) e^{-\sum_i \theta_i'^* b_i \theta_i} = \prod_i b_i = \det B$$

$$\boxed{\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij}\theta_j} = \det B}$$

5.6.2. Dirac Propagator

remember Dirac spinor $\hat{\Psi} = (\hat{\Psi}_1 \; \hat{\Psi}_2 \; \hat{\Psi}_3 \; \hat{\Psi}_4)^T$; now it becomes

$$\Psi(x) = \sum_i \Psi_i \, \phi_i(x)$$

Grassmann number — ordinary Dirac spinor basis

i.e.: $\phi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \dots, \phi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

we can now evaluate:

$$\langle 0 | T\,\Psi(x_1)\,\bar{\Psi}(x_2) | 0 \rangle = \frac{\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, e^{iS}\,\Psi(x_1)\bar{\Psi}(x_2)}{\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \, e^{iS}}$$

Ex | with $S = \int d^4x\, \bar\Psi (i\cancel{\partial} - m)\Psi$

$$\boxed{\langle 0| T\Psi(x_1)\bar\Psi(x_2)|0\rangle = S_F(x_1-x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x_1-x_2)}}{\cancel{k} - m + i\varepsilon}}$$

Note: higher correlation functions are obtained from Wick's theorem

5.6.3. Generating Fuction

$$Z[\bar\eta, \eta] = \int \mathcal{D}\bar\Psi \mathcal{D}\Psi \exp\left[iS + \int d^4x (\bar\eta\Psi + \bar\Psi\eta)\right]$$

Grassmann valued source field

$$= Z_0 \cdot \exp\left[-\int d^4x\, d^4y\, \bar\eta(x) S_F(x-y)\eta(y)\right]$$

and therefore

$$\langle 0|T\Psi(x_1)\bar\Psi(x_2)|0\rangle = Z_0^{-1}\left(-i\frac{\delta}{\delta\bar\eta(x_1)}\right)\left(i\frac{\delta}{\delta\eta(x_2)}\right) Z[\bar\eta,\eta]\Big|_{\eta,\bar\eta=0}$$

5.7. Symmetries in the Path Integral

three point correlator for free scalar theory

$$\langle 0|T\phi_1\phi_2\phi_3|0\rangle = Z_0^{-1}\int \mathcal{D}\phi\, e^{iS}\,\phi_1\phi_2\phi_3$$

$$S = i\int d^4x \left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2\right]$$

now shift $\phi(x) \to \phi'(x) = \phi(x) + \varepsilon(x)$; leaves $\mathcal{D}\phi$ invariant, $\mathcal{D}\phi = \mathcal{D}\phi'$

$$0 = \int \mathcal{D}\phi\, e^{iS}\Big[i\int d^4x\, \varepsilon(-\partial^2 - m^2)\phi \cdot \phi_1\phi_2\phi_3 + \varepsilon_1\phi_2\phi_3 + \phi_1\varepsilon_2\phi_3 + \phi_1\phi_2\varepsilon_3\Big]$$

$$(\partial^2 + m^2)\langle 0|T\phi\,\phi_1\phi_2\phi_3|0\rangle = -i\delta(x-x_1)\langle 0|T\phi_2\phi_3|0\rangle + \text{cycl.}$$

in particular: $(\partial^2 + m^2)\langle 0|T\phi\phi_1|0\rangle = -i\delta(x-x_1)$

We see: Feynman propagator is the Green's function of the classical field equations.

This idea generalises to arbitrary Lagrangians $\mathcal{L}$ (as long as the measure is preserved under shifts)

$$\boxed{\left\langle \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial \phi_\mu)} \right) \right] \phi_1 \cdots \phi_n \right\rangle = \sum_{i=1}^{n} \langle \phi_1 \cdots (i\delta(x-x_i)) \cdots \phi_n \rangle}$$

Schwinger - Dyson equation contact term