4.3. The relativistic point particle

Question: What is the simplest relativistic model?

$\rightsquigarrow$ point particle not subject to froces (free)

Remember the non-rel. version:

$$S = \int dt\, L = \int dt\, \tfrac{1}{2} \overbrace{m}^{\text{mass}} v^2(t) \qquad v^2 = \vec{v}\cdot\vec{v}, \quad \vec{v} = \dot{\vec{x}}$$

e.o.m: $$\delta S = 0 = \int dt\, m v \frac{d}{dt}\delta x(t) \overset{\text{i.b.p}}{=} -\int dt\, \underbrace{m\cdot\dot{v}}_{=0}\, \delta x(t)$$

free

$$\rightsquigarrow \boxed{m\cdot\dot{v} = m\cdot a = 0} \qquad \text{remember } m\cdot a = F \overset{\downarrow}{=} 0$$

Free particle moves with constant velocity.

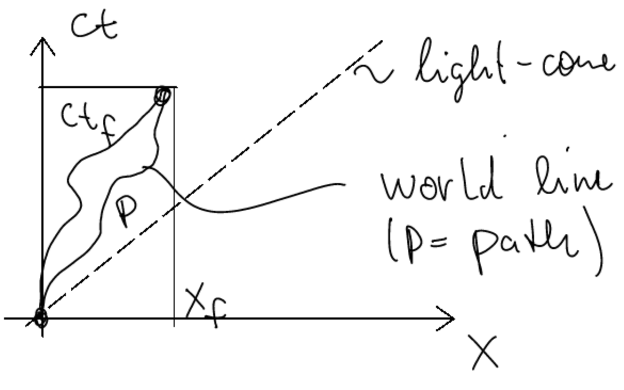
relativistic version?

Action and e.o.m. have to be invariant under Lorentz transformations!

$\rightsquigarrow$ implements 1. and 2. from last lecture

$m\dot{v} = 0$ is inv. under $\vec{v} \to \vec{v} + \underbrace{\vec{v}_0}_{\text{const.}}$

but S is not



Which quantity all observers agree on?

$\rightsquigarrow$ proper time on P

Proper time = time τ measured by a clock moving a long the path P

$$-ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

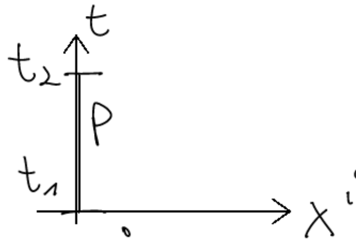
units: $[ds] = L$ ~ lenght $\qquad [c] = \frac{L}{T}$ ~ time

$\rightsquigarrow \quad [ds/c] = T$

$$\boxed{\tau = \int_P \frac{ds}{c}}$$

in rest frame:

$t = t, \quad x^i = 0 \rightsquigarrow dx^i = 0$



$$\tau = \int_{t_1}^{t_2} dt = t_2 - t_1 \checkmark$$

good candidate for the action but wrong

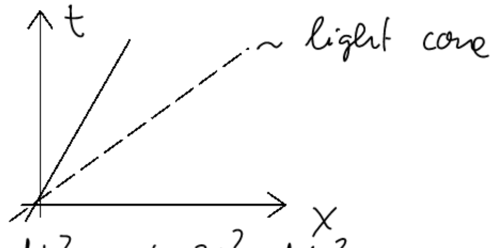
unit $\quad [S] = M \cdot \frac{L^2}{T^2} \cdot T \quad$ (from $\int dt \frac{1}{2} m v^2$)

$$= ML^2/T$$

$S = m \cdot (\text{velocity})^2 \cdot \tau \quad$ which velocity? c! because invariant

$$\rightsquigarrow \quad \boxed{S = -m \cdot c \int_P ds}$$

to understand - sign, look at Lorentz observer



$\vec{x}(t) = \vec{v}(t), \quad t = t$

$= (x^1\ x^2\ x^3)^T = (v^1\ v^2\ v^3)$

$$ds^2 = c^2 \cdot dt^2 - (v^1)^2 dt^2 - (v^2)^2 dt^2 - (v^3)^2 dt^2$$

$$= c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$S = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1-\frac{v^2}{c^2}} = \int_{t_1}^{t_2} dt \; L \quad \text{with}$$

Lagrangian $\boxed{L = -mc^2 \sqrt{1-\frac{v^2}{c^2}}}$

Check: limit $v \ll c$ with Taylor expansion

$$L = \underbrace{-mc^2}_{\text{const. (not important for e.o.m.)}} + \underbrace{\frac{1}{2}mv^2}_{\text{non-relativistic } L \checkmark} + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Hamiltonian: 1) canonical momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} \qquad \text{trick, write } v^2 = \vec{v}\cdot\vec{v}$$

$$\vec{v} = \dot{\vec{x}}$$

$$H = \vec{p}\cdot\vec{v} - L = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \text{energy of relativist particle}$$

for $v = 0$ $\quad E = mc^2$:-)

Symmetries of L: 1) Lorentz (by construction)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

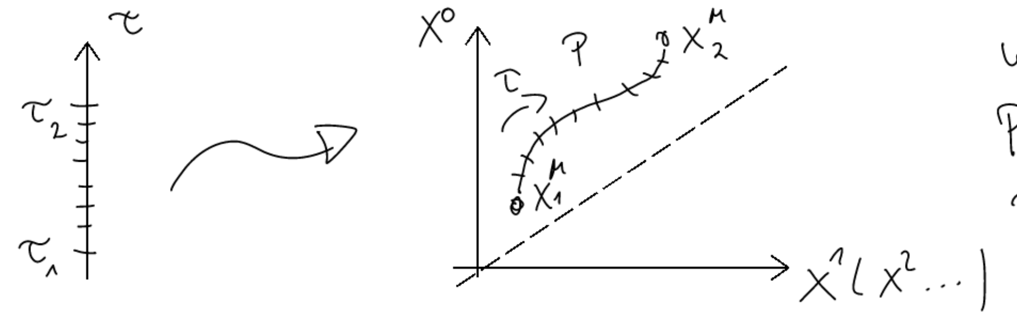
$+$

2) translations $x'^\mu = x^\mu + a^\mu$

$\stackrel{!}{=}$ Poincaré group we already know from KG

3) Reparameterization invariance

$$x^\mu = x^\mu(\tau) \qquad \text{with} \qquad x^\mu_{1/2} = x^\mu(\tau_{1/2})$$



world line P parameterized by τ

$$ds^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} (d\tau)^2$$

What happens if we change parameter from τ to τ' ?

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'} \frac{d\tau'}{d\tau}$$

$$ds^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau'} \frac{dx^\nu}{d\tau'} (d\tau')^2 = ds'^2$$

therefore the action is independent of the param.

$$S = -mc \int_{\tau_1}^{\tau_2} ds = -mc \int_{\tau_1'}^{\tau_2'} ds'$$

Equations of motion:

$$\delta S = -mc \int_{\tau_1}^{\tau_2} \delta(ds) \quad \text{with}$$

$$\delta(ds^2) = 2\, ds\, \delta(ds) = -2\eta_{\mu\nu}\, d(\delta x^\mu)\, dx^\nu$$

$$\Rightarrow \delta(ds) = -\eta_{\mu\nu}\, d(\delta x^\mu) \frac{dx^\nu}{ds} \quad \text{, Parameterized by } \tau$$

$$= -\eta_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{ds} \cdot d\tau$$

$$\delta S = mc \int_{\tau_1}^{\tau_2} \eta_{\mu\nu} \frac{d}{d\tau} \delta x^\mu \frac{dx^\nu}{ds} d\tau$$

$$\delta S \overset{\text{i.b.p}}{=} -mc \int_{\tau_1}^{\tau_2} \eta_{\mu\nu} \, \delta x^\mu \frac{d}{d\tau}\frac{dx^\nu}{ds} \overset{!}{=} 0$$

we define $u^\nu := c \cdot \frac{dx^\nu}{ds}$ – change of proper time

four-velocity

$m \cdot u^\nu = p^\nu$ ~ four-momentum

e.o.m: $\boxed{\frac{dp^\mu}{d\tau} = 0}$

EX 6: much more on four-velocity & acceleration