Last lecture: We encountered the path integral. In particular, we related it to the Feynman propagator

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left[i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}\right]}{\int \mathcal{D}\phi \exp\left[i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}\right]} = \frac{I_2}{I_1}$$

required to normalise vacuum state $\langle 0|0\rangle = 1$

5.3. Wick's Theorem and Feynman Diagrams

Idea: compute RHS directly for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$

like last lecture discretise in box

$$\phi(x_i) = \frac{1}{L^4}\sum_n e^{-ik_n x_i}\phi(k_n), \quad k_n^\mu = \frac{2\pi n^\mu}{L}$$

$$|n^\mu| < \frac{\pi}{\varepsilon}$$

$\phi(x)$ is real $\leadsto$ $\phi^*(k) = \phi(-k)$ or $\mathrm{Re}/\mathrm{Im}[\phi(k_n)]$ as independent variables for $k_n^0 > 0$

$$\mathcal{D}\phi(x) = \prod_{k_n^0>0} d\,\mathrm{Re}\,\phi(k_n)\, d\,\mathrm{Im}\,\phi(k_n), \text{ and}$$

$$S = \int d^4x\, \mathcal{L} = \lim_{\substack{L\to\infty\\ \varepsilon\to 0}}\left[\frac{1}{L^4}\sum_{k_n^0>0}(m^2 - k_n^2)|\phi_n|^2\right] \quad \phi_n = \phi(k_n)$$

$$I_1 = \int \mathcal{D}\phi\, e^{iS} = \prod_{k_n^0>0}\int d\,\mathrm{Re}(\phi_n)\exp\left[\frac{i}{L^4}(m^2-k_n^2)\mathrm{Re}^2\phi_n\right]$$

$$\times \int d\,\mathrm{Im}(\phi_n)\exp\left[\frac{i}{L^4}(m^2-k_n^2)\mathrm{Im}^2\phi_n\right]$$

$$= \prod_{\text{all } k_n}\sqrt{\frac{-i\pi L^4}{m^2-k_n^2}} \quad \text{after } L\to\infty \;\&\; \varepsilon\to 0$$

Gaussian integral, see EX.

in particular $\left(\prod_k \int d\xi^k\right) \exp\left[-\xi^i A_{ij} \xi^j\right]$

$$= \prod_i \left(\int dx^i \exp\left[-a_i (x^i)^2\right]\right) = \prod_i \sqrt{\frac{\pi}{a_i}}$$

eigenvectors of A with eigenvalue a_i

$$= \text{const} \times \left[\det A\right]^{-1/2}$$

$S = \frac{1}{2}\int d^4x \, \phi\left(-\partial^2 - m^2\right)\phi$ after i.b.p., thus

$$\int \mathcal{D}\phi \, e^{iS} = \text{const} \times \left[\det\left(m^2 + \partial^2\right)\right]^{-1/2}$$

For I_2 we need:

$$\phi(x_1)\,\phi(x_2) = \frac{1}{L^8}\sum_{m,l} e^{-i(k_m x_1 + k_l x_2)} \phi_m \phi_l$$

$$I_2 = \ldots = \frac{1}{L^8}\sum_m e^{-ik_m\cdot(x_1 - x_2)} \underbrace{\left(\prod_{k_n^0>0} \frac{-i\pi L^4}{m^2 - k_n^2}\right)}_{I_1} \frac{-iL^4}{m^2 - k_m^2 - i\varepsilon}$$

Therefore we obtain after the continuum limit $L\to\infty$, $\varepsilon\to 0$

finally the expected result:

$$\langle 0 | T\hat\phi(x_1)\,\hat\phi(x_2)|0\rangle = \lim_{\varepsilon\to 0^+}\int \frac{d^4k}{(2\pi)^4}\frac{i e^{-ik(x_1-x_2)}}{k^2 - m^2 + i\varepsilon}$$

$$= D_F(x_1 - x_2) \quad //$$

We can now also compute more complicated correlations like

cumbersome, use trick?

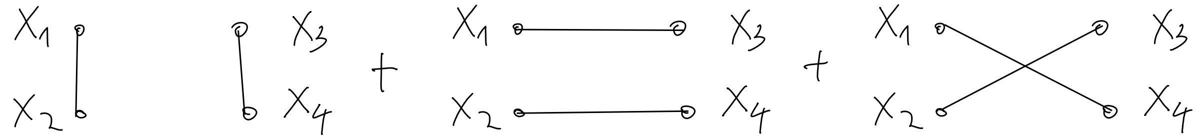
$$\langle 0 | T\hat\phi(x_1)\,\hat\phi(x_2)\,\hat\phi(x_3)\,\hat\phi(x_4)|0\rangle = \overset{\downarrow}{\ldots} =$$ Wick contraction

$$\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

Wick's Theorem: Add all possible contractions with the basic contraction:

$$\phi_i := \phi(x_i) \rightsquigarrow \underbrace{\phi_1 \; \phi_2} = D_F(x_1 - x_2)$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4) + D_F(x_1-x_4)\,D_F(x_2-x_3)$$

$=$ 

These diagrams are called Feynman diagrams.

5.4. Interacting Theories

Things become even more interesting with interacting Lagrangians:

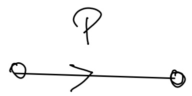
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \qquad \text{i.e. } \mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

$\mathcal{L}_0$: quadratic $\rightarrow$ solvable by Gaussian integrals

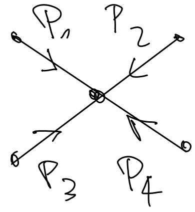
For small λ we can expand:

$$\exp\left[i\int d^4x\,\mathcal{L}\right] = \exp\left[i\int d^4x\,\mathcal{L}_0\right]\left(1 - i\int d^4x\,\frac{\lambda}{4!}\phi^4 + \dots\right)$$

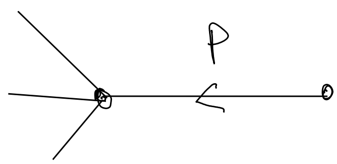
I_1 and I_2 can now be written in terms of

I) 

$= \dfrac{i}{p^2 - m^2 + i\varepsilon}$,

II) 

$= -i\lambda\,(2\pi)^4\,\delta^{(4)}\left(\sum_i p_i\right)$,

III) external line 

$= e^{-ipx}$

IV) integrate over each underdetermined loop momentum: $\int \frac{d^4p}{(2\pi)^4}$

V) incoorporate symmetry factors

5.5. Functional Derivatives and Generating Functions

more elegant way to evaluate path integrals:

use <u>generating function</u>

$$\boxed{Z[J] = \int \mathcal{D}\phi \exp\left[i\int d^4x \left(\mathcal{L} + J(x)\phi(x)\right)\right]}$$

source term

$$\langle 0|T\,\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\dots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

with $Z_0 = Z[J=0]$

for the free theory:

$$Z[J] = Z_0 \exp\left[-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right]$$

↗ EX derive Wick's theorem using this generating function.