4. Representation theory

4.1. Adjoint representation

Def.: Let G be a Lie group and $g, h \in G$, then the <u>adjoint action</u> is defined as

$$\boxed{Ad_g(h) = g \cdot h \cdot g^{-1}}$$

It is a group homomorphism $g \mapsto Ad_g$:

$$Ad_{g_2} \circ Ad_{g_1}(h) = Ad_{g_2}(g_1 h g_1^{-1}) = g_2 g_1 h g_1^{-1} g_2^{-1}$$
$$= Ad_{g_2 \cdot g_1}$$

composition $\circ$ becomes the group multiplication!

around the identity (remember exp-map)

$h = 1 + X + \dots$ $\qquad$ $\left.\begin{array}{l} g = 1 + Y + \dots \\ g^{-1} = 1 - Y + \dots \end{array}\right\}$ - higher orders

$X, Y \in \mathfrak{g}$ ← Lie algebra of G

$$Ad_g(h) = (1 + Y + \dots)(1 + X + \dots)(1 - Y + \dots)$$
$$= 1 + [Y, X] + \dots$$

⇝ for the Lie algebra $\boxed{ad_Y(X) = [Y, X]}$

again homomorphism:

$$[ad_{X_2}, ad_{X_1}](Y) = (ad_{X_2} \circ ad_{X_1} - ad_{X_1} \circ ad_{X_2})(Y) =$$
$$ad_{X_2}([X_1, Y]) - ad_{X_1}([X_2, Y]) = [X_2, [X_1, Y]] - [X_1, [X_2, Y]]$$
$$= [[X_2, X_1], Y] = ad_{[X_1, X_2]}(Y)$$

Jacobi identity

All informations for $ad_X(Y)$ are contained in the Lie algebra's structure coefficients.

$$ad_{T_a}(T_b) = [T_a, T_b] = \sum_c f_{ab}{}^c T_c \quad \text{(see sec. 2.4)}$$

$ad_{T_a}(\cdot)$ is a representation, we find for any Lie algebra g:
$$ad_{T_a}(T_b) = \sum_{c = 1, \dots, \dim G} (f_a)_b{}^c T_c$$

$(f_a)_b{}^c$ are $\dim G$ different $(\dim G) \times (\dim G)$-matrices

4.2. Killing form

There is a natural inner product on the Lie algebra called the Killing form (or metric)

$$K(X,Y) = \frac{1}{I} \operatorname{tr}(ad_X \circ ad_Y)$$

normalization constant

trace in the adjoint matrix representation

for example matrix $\rho_a{}^b$, $\operatorname{tr} \rho = \sum_a \rho_a{}^a$

therefore:

$$ad_{T_a} \circ ad_{T_b}(T_c) = ad_{T_a}([T_b, T_c])$$
$$= \sum_d f_{bc}{}^d \, ad_{T_a}(T_d) = \sum_{d,e} \underbrace{f_{bc}{}^d f_{ad}{}^e}_{(\rho_{ab})_c{}^e} T_e$$

$$\leadsto \boxed{K(T_a, T_b) = \mathcal{K}_{ab} = \frac{1}{I} \sum_{c,d} f_{ad}{}^c f_{bc}{}^d}$$

Properties: • $K(X,Y) = K(Y,X)$ symmetric

• $K([X,Y],Z) + K(Y,[X,Z]) = 0$ invariant under adjoint action

4.3. Semi-simple Lie algebras

Def.: Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. $\mathfrak{h}$ is called an idea if $\forall x \in \mathfrak{g}, y \in \mathfrak{h}$ $[X,Y] \in \mathfrak{h}$ holds.

We also define:

> (I.) An algebra is called simple if it does not have any proper (strictly smaller than the full algebra) ideals.

> (II.) An algebra is called semi-simple if it is the direct sum of simple algebras.

Example: $U \in U(N) \rightsquigarrow U^{+} = U^{-1}$

$U = \exp(i\delta u) \rightsquigarrow \delta u^{+} = \delta u \quad (\mathbb{1}^{+} = \mathbb{1})$

$\delta u = \mathbb{1}$ // all other traceless, hermitian matrices

check $[\mathbb{1},\mathbb{1}] = 0 \cdot \mathbb{1}$ subalgebra ✓

$[\mathbb{1},X] = 0 \cdot \mathbb{1} \rightsquigarrow$ ideal

$\rightsquigarrow \mathbb{1}$ is a proper ideal in the Lie algebra $u(N)$

$\rightsquigarrow u(N)$ is not simple

by removing this ideal, we get $su(N)$ which is simple.

Theorem: (Cartan) $\boxed{\text{A Lie algebra is \underline{semi-simple} iff (if and only if) the Killing form is non-singular.}}$

4.4. Solvable Lie algebras

Def.: The <u>derived Lie algebra</u> $[\mathfrak{g},\mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$ is the subalgebra that consists of all combinations of Lie brackets for pairs of elements of $\mathfrak{g}$.

Example: Heisenberg algebra with commutators:

$$[X,P]=I, \quad [X,I]=0, \quad [P,I]=0$$

$\rightsquigarrow$ derived Lie algebra $[\mathfrak{g},\mathfrak{g}] = \mathrm{span}(\{I\})$
$$\mathfrak{g} = \mathrm{span}(\{X,P,I\})$$

Def.: The <u>derived series</u> is the sequence of derived Lie algebras: $\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$

Def.: A Lie algebra $\mathfrak{g}$ is <u>solvable</u> if its derived series terminates in the zero subalgebra.

Heisenberg algebra with $\mathfrak{g}^{(1)} = \mathrm{span}(\{I\})$, $\mathfrak{g}^{(2)} = \{0\}$, is solvable!

Theorem: (Levi)

Any finite dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is the semidirect product of a solvable ideal and a semi-simple subalgebra