Last lecture: Three of four forces in nature are governed by Yang-Mills theory.

Knowing how to quantise it is paramount.

$\lightning$ Various challenges: 1. we are dealing with a gauge theory $\rightarrow$ unphysical degrees of freedom

2. loop effects lead to ∞'s which we have to regularise

$\hookrightarrow$ 3. dependence on energy scale $\rightarrow$ renormalisation

We will deal with them for the rest of the course.

Our most important tool = $\boxed{\text{PATH INTEGRAL}}$

5. Path Integral Formalism

5.1. Single Particle

Take a single particle moving in a potential $V(x)$ as warmup.

Hamiltonian: $$H = \frac{p^2}{2m} + V(x)$$

Question: What is the likelyhood that the particle travels in the time T from point x_a to x_b?

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle$$

Basic idea behind the path integral: Write this amplitude as a sum over different phases along all pathes from x_a to x_b.

$$U(x_a, x_b; T) = \sum_{\text{all pathes}} e^{i \cdot \text{phase}} = \int \mathcal{D}x(t)\, e^{i\,\text{phase}}$$

"sum" over ∞'ly many pathes is written as integral over the continious space of function $x(t)$.

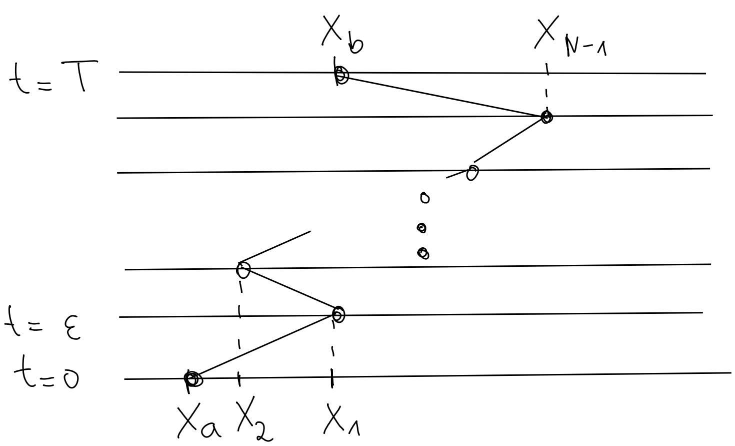
$$\text{phase} = \frac{S[x(t)]}{\hbar}$$

because then for $S \gg \hbar$ stationary phase approximation just gives the classical contribution $\delta S[x(t)] = 0$.

$$\boxed{U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{i S[x(t)]/\hbar}}$$

complicated integral
evaluate it by discretisation

$$S = \int_0^T dt \left(\frac{m}{2}\dot{x}^2 - V(x)\right) \to \sum_k \left[\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\varepsilon} - \varepsilon V\left(\frac{x_{k+1}+x_k}{2}\right)\right]$$



$$\int \mathcal{D}x(t) = \int \frac{dx_{N-1}}{C(\varepsilon)}$$

$$\vdots$$

$$\int \frac{dx_2}{C(\varepsilon)} \cdot \int \frac{dx_1}{C(\varepsilon)} \cdot \frac{1}{C(\varepsilon)}$$

constant we still have to fix

$$\int \mathcal{D}x(t) = \frac{1}{C(\varepsilon)} \prod_k \int_{-\infty}^{\infty} \frac{dx_k}{C(\varepsilon)}$$

(I.) Just the last step gives

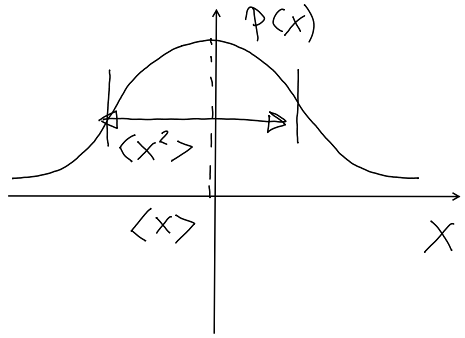
$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\varepsilon)} \exp\left[\frac{i}{\hbar}\frac{m(x_b - x')^2}{2\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x_b + x'}{2}\right)\right] \cdot U(x_a, x'; T-\varepsilon)$$

expand around $x' = x_b$

$$U(x_a,x_b;T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\varepsilon)} \exp\left(\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b-x')^2\right)\left[1-\frac{i\varepsilon}{\hbar}V(x_b)+\dots\right]$$
$$\cdot \left[1+(x'-x_b)\frac{\partial}{\partial x_b}+\frac{1}{2}(x'-x_b)^2\frac{\partial^2}{\partial x_b^2}+\dots\right]U(x_a,x_b;T-\varepsilon)$$

Gaussian integrals: $\int dx\, f(x)\, e^{-bx^2} = \langle f(x)\rangle$

$\langle 1\rangle = \sqrt{\frac{\pi}{b}}$, $\langle x\rangle = 0$, $\langle x^2\rangle = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$



results in:

$$U(x_a,x_b;T) = \frac{1}{C(\varepsilon)}\overbrace{\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}^{\overset{!}{=}\,C(\varepsilon)} \cdot$$
$$\Big(1 \underbrace{-\frac{i\varepsilon}{\hbar}V(x_b) + \frac{i\varepsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2}}_{\varepsilon H} + O(\varepsilon^2)\Big)\, U(x_a,x_b;T-\varepsilon)$$

or

$$i\hbar\frac{\partial}{\partial T}U(x_a,x_b;T) = H\,U(x_a,x_b;T) \quad \text{Schrödinger equation}$$

This derivation is very powerful and also works for more complicated systems with more degrees of freedom.

$$U(\vec{q}_a,\vec{q}_b;T) = \left(\prod_i \int \mathcal{D}q_i(t)\,\mathcal{D}p_i(t)\right)\exp\left[i\int_0^T dt\left(\sum_i p^i\dot{q}^i - H(q^i,p^i)\right)\right]$$

integral over phase space

→ apply it to the Klein-Gordon field

5.2. Klein-Gordon Field

remember: $H = \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \overset{m\phi^2}{\overset{\|}{V(\phi)}}\right]$

$$\langle\phi_b(\vec{x})|\,e^{-iHT}\,|\phi_a(\vec{x})\rangle = \int\mathcal{D}\phi\,\mathcal{D}\pi\,\exp\left[i\int_0^T d^4x\,(\pi\dot{\phi} - H)\right]$$

quadratic in π, use Gaussian integral to compute $\int\mathcal{D}\pi$

↗ EX 4

$$\boxed{\langle \phi_b | e^{-iHT} | \phi_a \rangle = \int \mathcal{D}\phi \, \exp\left[i \int_0^T d^4x \, \mathcal{L} \right]}$$

with $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$

Let us now try to rederive the propagator

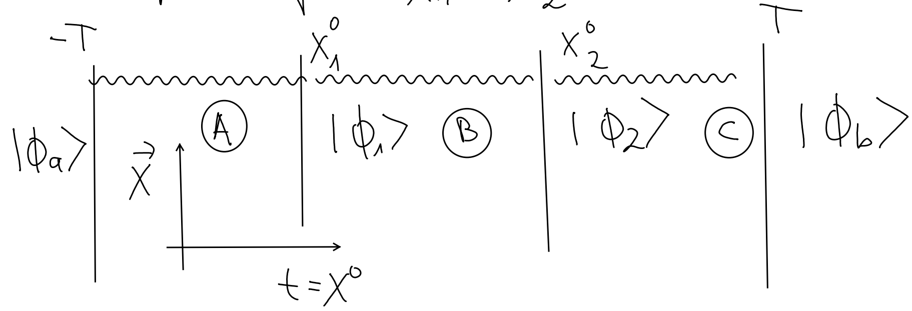
$\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle$ from the path integral.

remember here $\hat{\phi}(x)$ are operators on the Hilbert space while in the path integral $\phi(x)$ are plain functions.

→ we look at the integral

$$\underbrace{\int \mathcal{D}\phi(x)}_{\int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\substack{\phi(x_1^0, \vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^0, \vec{x}) = \phi_2(\vec{x})}}} \phi(x_1)\, \phi(x_2) \, \exp\left[i \int_{-T}^{T} d^4 \mathcal{L}(\phi) \right]$$

time ordering is by definition build into the path integral and we find for $x_1^0 < x_2^0$:



$$\stackrel{!}{=} \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \; \phi(\vec{x}_1)\, \phi(\vec{x}_2) \underbrace{\langle \phi_b | e^{-iH(T - x_2^0)} | \phi_2 \rangle}_{C}$$

$$\times \underbrace{\langle \phi_2 | e^{-iH(x_2^0 - x_1^0)} | \phi_1 \rangle}_{B} \underbrace{\langle \phi_1 | e^{-iH(x_1^0 + T)} | \phi_a \rangle}_{A}$$

$+ \; (x_1^0 \Leftrightarrow x_2^0)$ for $x_2^0 < x_1^0$

completeness relation: $\int \mathcal{D}\phi_1 \, |\phi_1\rangle \langle \phi_1| = \mathbb{1}$ (same for ϕ_2)

$$\overset{!}{=} \langle \phi_b | e^{-iHT} \, T\left[\hat{\phi}(x_1)\, \hat{\phi}(x_2)\right] e^{-iHT} | \phi_a \rangle$$

remember $\hat{\phi}(x) = e^{iHx^0} \phi(\vec{x}) \, e^{-iHx^0}$

last step: $T \to \infty(1 - i\varepsilon)$ ← for convergence

then
$$e^{-iHT} | \phi_a \rangle \sim |0\rangle$$
$$e^{iHT} | \phi_b \rangle \sim |0\rangle$$
} all excitations decay in the infinit past / future.

Success! We found the Feynman propagator from the path integral.