3.2. Maurer-Cartan form

the forms e^a and their dual vector fields can be used to construct general left-inv. tensor fields

$$T = T^{a_1 \dots a_m}_{b_1 \dots b_n} \, e^{b_1} \otimes \dots \otimes e^{b_n} \otimes e_{a_1} \otimes \dots \otimes e_{a_m}$$

Question: Is there an effective way to construct them?

Def: If V is a vector space over the field $\mathbb{F}$, the general linear group $GL(V)$ is the group of all automorphism of V. bijective linear maps
The functional composition acts as group multiplication.

for physicists: $GL(n, \mathbb{F})$ is the set of all
because bijective ~ invertible $n \times n$ matrices with entries in $\mathbb{F}$.

Def.: Given a group G with multiplication $\circ$, a subset H of G is called a subgroup if H also forms a group under $\circ$.

Def.: Subgroups of $GL(n, \mathbb{F})$ are called matrix groups. ~ because their elements can be written as $n \times n$ matrices.

Remember $SO(3)$ example!

For a matrix group G, we define the

left-invariant <u>Maurer-Cartan form</u>

$$\boxed{e = g^{-1} dg}$$

properties:

1. left-invariant because $\phi^* df = d(\phi^* f)$

$$L_h^* e = (L_h^* g^{-1}) \tilde{d} (L_h^* g) \qquad dh = 0$$
$$= (hg)^{-1} d(hg) = g^{-1} h^{-1} h dg$$
$$= g^{-1} dg = e$$

2. satisfies the <u>Maurer-Cartan equation</u>

$$\boxed{de + e \wedge e = 0}$$

$$de = d(g^{-1} dg) = dg^{-1} dg$$
$$= dg^{-1} g g^{-1} dg$$
$$= -g^{-1} dg \wedge g^{-1} dg = -e \wedge e \qquad \square$$

We can parametrize e by

$$e = T_a e^a{}_i \, dx^i \qquad i = 1, \dots, \dim G$$
$$a = 1, \dots, \dim G$$
$$= T_a e^a$$

$$d(T_a e^a) = T_a de^a = -T_b \cdot T_c e^b \wedge e^c$$
$$= -\tfrac{1}{2} [T_b, T_c] e^b \wedge e^c$$

remember from the def. of Lie algebra

$$[T_a, T_b] = f_{ab}{}^c T_c$$

$$T_a \left(d e^a + \tfrac{1}{2} f_{bc}{}^a e^b \wedge e^c \right) = 0$$

left-inv. one form on Lie group G $\longrightarrow$ structure constants of Lie algebra $\mathfrak{g}$

3.3. Exponential map

Question: Is the opposite

$$\text{Lie algebra } \mathfrak{g} \xrightarrow[\exp]{} \text{Lie group } G$$

also possible?

YES! With exponential map, $\exp : \mathfrak{g} \to G$

Consider 1-parameter subgroup $\gamma^X(t)$ of G, generated by $X \in \mathfrak{g}$. We have:

$$\gamma^X(0) = e, \quad \gamma^X(t+s) = \gamma^X(t) \overset{\text{group mult.}}{\cdot} \gamma^X(s), \quad \dot{\gamma}^X(0) = X$$

Def.: $\exp X = \gamma^X(1)$ and with

$$\gamma^X(k \cdot t) = \gamma^{kX}(t) \quad \text{because} \quad \frac{d}{dt} \gamma^X(k \cdot t)\Big|_{t=0} = k \cdot X$$

$$\boxed{\gamma^X(t) = \exp(t \cdot X)}$$

For matrix groups this is easier:

$$\exp X = 1 + X + \tfrac{1}{2}X^2 + \dots = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

⚡ Only component of Lie group connected to identity. For $O(n)$ remember $\det g = \pm 1$, but $\det(\exp X) = e^{\operatorname{tr} X} = \underbrace{e^0}_{\text{because } X^T = -X} = 1$.

3.4. Classical matrix groups

We already have seen the constraint

$\det g = 1 \leadsto$ implies that g is invertible

results in $\underline{SL(n, \underbrace{\mathbb{F}) \text{ special linear group}}_{\mathbb{R} \text{ or } \mathbb{C}}}$

⚡ $\det(A \cdot B) \neq \det A \cdot \det B$ for quaternions

Furthermore a $\overset{\text{non-singular}}{\text{metric}}$ on V of $GL(V)$ can be preserved.

symmetric

$$h^\vee = \begin{pmatrix} \mathbb{1}_{N_+} & 0 \\ 0 & -\mathbb{1}_{N_-} \end{pmatrix}$$

orthogonal group

$O(N_+, N_-, \mathbb{F})$

antisymmetric

$$h^\wedge = \mathbb{1}_N \otimes \varepsilon \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symplectic group

$Sp(2N, \mathbb{F})$

for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$ we can also have the unitary group $U(N_+, N_-, \mathbb{F})$