4.2. Local symmetries

local: transformation can be different on different points of the worldsheet

① Reparametrisation (= diffeomorphism)

structure preserving map from one differentiable manifold to another

$$\tilde{\sigma}^\alpha = \sigma^\alpha - \xi^\alpha(\sigma)$$

n-vector, generates diffeomorphism

How are the constituents of the action transforming?

i) world-sheet scalar: $X^\mu(\sigma) = X^\mu(\tilde{\sigma} + \xi)$

$$X^\mu = \tilde{X}^\mu + \xi^\alpha \tilde{\partial}_\alpha \tilde{X}^\mu$$

ii) partial derivative:

$$\partial_\alpha = \frac{\partial}{\partial \sigma^\alpha} = \frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^\alpha}\frac{\partial}{\partial \tilde{\sigma}^\beta} = \left(\delta^\beta_\alpha - \partial_\alpha \xi^\beta\right)\tilde{\partial}_\beta$$

iii) derivative on scalar i) ⇝ one-form

$$\begin{aligned}\partial_\alpha X^\mu &= \partial_\alpha\left(\tilde{X}^\mu + \xi^\beta \tilde{\partial}_\beta \tilde{X}^\mu\right)\\ &= \partial_\alpha \tilde{X}^\mu + \partial_\alpha \xi^\beta \tilde{\partial}_\beta \tilde{X}^\mu + \xi^\beta \tilde{\partial}_\beta \partial_\alpha \tilde{X}^\mu\\ &= \partial_\alpha \tilde{X}^\mu + \underbrace{\partial_\alpha \xi^\beta \partial_\beta \tilde{X}^\mu + \xi^\beta \partial_\beta \partial_\alpha \tilde{X}^\mu}_{= L_\xi(\partial_\alpha \tilde{X}^\mu) = \delta(\partial_\alpha X)} + \mathcal{O}(\xi^2)\end{aligned}$$

Lie derivative on one-form ω_α: $\boxed{L_\xi \omega_\alpha = \xi^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta}$

from i) ⇝ on scalar φ: $\boxed{L_\xi \varphi = \xi^\alpha \partial_\alpha \varphi}$

Lie derivative satisfies Leibniz rule

$$L_\xi(\omega_\alpha v^\alpha) = \xi^\beta \partial_\beta(\omega_\alpha v^\alpha) = (L_\xi \omega_\alpha)v^\alpha + \omega_\alpha L_\xi v^\alpha$$

vector

→ for a vector v^α $\boxed{L_\xi v^\alpha = \xi^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta \xi^\alpha}$

Similar for higher rank tensors, like the metric $h_{\alpha\beta}$

$$\delta h_{\alpha\beta} = L_\xi h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + 2 h_{\gamma(\alpha} \partial_{\beta)} \xi^\gamma$$

or

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\; h^{\alpha\beta}\,\delta h_{\alpha\beta}$$
$$= \frac{1}{2}\sqrt{-h}\left(\xi^\gamma(\partial_\gamma h_{\alpha\beta}) h^{\alpha\beta} + 2\,\partial_\alpha \xi^\alpha\right)$$
$$= \xi^\alpha \partial_\alpha \sqrt{-h} + \sqrt{-h}\,\partial_\alpha \xi^\alpha = \partial_\alpha\left(\sqrt{-h}\,\xi^\alpha\right)$$

for a weight 1 scalar density

trick from GR: Substitute ∂_α (partial derivative) in Lie derivative with ∇_α (covariant derivative) will not change result ↗ EX

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① Covariant derivative

$$\nabla_\alpha T^{\beta_1\ldots\beta_m}{}_{\gamma_1\ldots\gamma_n} = \partial_\alpha T^{\beta_1\ldots\beta_m}{}_{\gamma_1\ldots\gamma_n} + \Gamma^{\beta_1}{}_{\delta\alpha} T^{\delta\ldots\beta_m}{}_{\gamma_1\ldots\gamma_n}$$
$$\ldots + \Gamma^{\beta_m}{}_{\delta\alpha} T^{\beta_1\ldots\delta}{}_{\gamma_1\ldots\gamma_n} - \Gamma^{\delta}{}_{\gamma_1\alpha} T^{\beta_1\ldots\beta_m}{}_{\delta\ldots\gamma_n} - \Gamma^{\delta}{}_{\gamma_n\alpha} T^{\beta_1\ldots\beta_m}{}_{\gamma_1\ldots\delta}$$

② Christoffel symbols

$$\Gamma^\alpha{}_{\beta\gamma} = 1/2\, h^{\alpha\delta}\left(\partial_\gamma h_{\delta\beta} + \partial_\beta h_{\delta\gamma} - \partial_\delta h_{\beta\gamma}\right)$$

Why? Because $L_\xi(\partial_\alpha w_\beta) \neq \partial_\alpha(L_\xi w_\beta)$, but
$L_\xi(\nabla_\alpha w_\beta) = \nabla_\alpha(L_\xi w_\beta)$

③ Riemann tensor

$$[\nabla_\alpha, \nabla_\beta] V^\gamma = R_{\alpha\beta\delta}{}^{\gamma} V^\delta$$

Ricci tensor: $R_{\alpha\beta} = R_{\gamma\alpha\beta}{}^{\gamma}$; Ricci scalar $R = R_\alpha{}^\alpha$

But $\nabla_\alpha h_{\beta\gamma} = 0$ (metric compatibility) $\nearrow$ EX

$\leadsto \quad \delta h_{\alpha\beta} = 2\nabla_{(\alpha}\xi_{\beta)}$

conserved currents

$$\delta S_P = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\left(T_{\alpha\beta}\,\delta h^{\alpha\beta} + 0\right)$$ on-shell for X^μ, i.e. $\delta X^\mu = 0$

$$= -T\int d^2\sigma \sqrt{-h}\left(T_{\alpha\beta}\nabla^\alpha \xi^\beta\right)$$

$$= +T\int d^2\sigma \sqrt{-h}\left(\underbrace{\nabla^\alpha T_{\alpha\beta}}_{=0}\xi^\beta\right) = 0$$

$\leadsto$ $T_{\alpha\beta}$ is conserved current for WS diffeom. $\boxed{\nabla_\alpha T^\alpha{}_\beta = 0}$

$$P_\beta = \oint_0^{2\pi} d\sigma\, T^0{}_\beta\,,$$ conserved charge $=$ world sheet momentum

2. Conformal invariance

Remember EX: $h_{\alpha\beta} \to e^{2\lambda(\sigma)} h_{\alpha\beta} = h_{\alpha\beta} + \delta h_{\alpha\beta}$

$$\delta h_{\alpha\beta} = 2\lambda(\sigma)\, h_{\alpha\beta}$$

and $\quad \delta X^\mu = 0$

only in two dimensions! For n-dim world surface:

$\sqrt{-h} \to (e^{2\lambda})^{n/2}\sqrt{-h}$ and

$h^{\alpha\beta} \to (e^{2\lambda})^{-1} h^{\alpha\beta}$

$$\int d^n\sigma \sqrt{-h}\, h^{\alpha\beta} G_{\alpha\beta} \to (e^{2\lambda})^{\overbrace{n/2-1}^{0}} \int d\sigma^n \sqrt{-h}\, h^{\alpha\beta} G_{\alpha\beta}$$

$\overbrace{\qquad\qquad}^{1}$ only for $\boxed{n=2}$

combining 1. diffeom. and 2. conformal transform.

$$\boxed{\delta h_{\alpha\beta} = \underbrace{2\nabla_{(\alpha}\xi_{\beta)} - \nabla^{\gamma}\xi_{\gamma} h_{\alpha\beta}}_{\text{trace-free}} + 2(\lambda + 1/2\, \nabla^{\gamma}\xi_{\gamma})\, h_{\alpha\beta}}$$

if we set this part to 0 we can show that

$\nabla_\alpha T^\alpha{}_\beta = 0$ implies $\nabla_\alpha (T^\alpha{}_\beta \xi^\beta) = 0$

$$= (\overset{=0}{\nabla_\alpha T^\alpha{}_\beta})\xi^\beta + T^{\alpha\beta}\nabla_\alpha \xi_\beta = \frac{1}{2}\underbrace{T^{\alpha\beta} h_{\alpha\beta}}_{=0}\nabla_\gamma \xi^\gamma$$

$= 0$ because of conformal invariance (see EX)

ξ^α with these properties are called conformal Killing vectors

$\leadsto$ There are ∞ of them!

$\hookrightarrow$ In light cone gauge: $\sigma^{\pm} = \tau \pm \sigma$

(please check for yourself that) $h_{++} = 0$, $h_{+-} = \frac{1}{2}$, $h_{--} = 0$

$\Gamma_{\alpha\beta}{}^{\gamma} = 0$

conformal Killing vector is $\boxed{\xi^{\pm} = \xi^{\pm}(\sigma^{\pm})}$

check:

$$2\nabla_{(\alpha}\xi_{\beta)} - \nabla_\gamma \xi^\gamma h_{\alpha\beta} = 0 \begin{cases} 1/2(\partial_+ \xi^- + \partial_- \xi^+) = 0 \ \checkmark & \text{for } (\alpha,\beta) = (\pm,\pm) \\ 1/2(\partial_+ \xi^+ + \partial_- \xi^-) - 1/2(\partial_+\xi^+ + \partial_-\xi^-) = 0 \ \checkmark & \text{for } (\alpha,\beta) = (\pm,\mp) \end{cases}$$

corresponding conserved charges are:

$$L_\xi(\tau) = 2T\int_0^{2\pi} d\sigma\, \xi^+(\tau+\sigma)\, T_{++}$$

$$\bar{L}_\xi(\tau) = 2T\int_0^{2\pi} d\sigma\, \xi^-(\tau-\sigma)\, T_{--}$$

Expand $\xi^{\pm}(\sigma^{\pm})$ in Fourier modes

$$\xi^{\pm}(\sigma^{\pm}) = \sum_{m=-\infty}^{\infty} e^{im\sigma^{\pm}}\xi^{\pm}_m$$

for each mode $e^{im\sigma^\pm}$ we obtain L_m and $\bar{L}_m$ with:

$$\boxed{\begin{array}{l}\{L_m, L_n\} = -i(m-n)L_{m+n},\\ \text{same for } \bar{L}_m \text{ and } \{L_m, \bar{L}_n\} = 0\end{array}}$$

Classical Virasoro algebra

Remarks: – ∞-dim Lie algebra
– subalgebra by L_0, L_1 and $L_{-1} \hat{=} SL(2,\mathbb{R})$
↳ together with $\bar{L}_0, \bar{L}_1$ and $\bar{L}_{-1} \rightsquigarrow SO(3,1)$
conformal group in 2 dimensions

→ Conformal Field Theory = CFT

Application: revisit conformal gauge

again n-dim. world surface

$h_{\alpha\beta}$: $\frac{1}{2}n(n+1)$ d.o.f

Gauge : $n+1$ (diff. + Weyl)

$\left.\right\}$ $\underbrace{\frac{n}{2}(n+1) - (n+1)}_{\text{for } n=2 \;=\; 0}$ cannot be removed!

only in $n=2$ (for the string) we can completely gauge fix the world sheet metric

Next lecture we look @ the quantisation!