2.4. Linear algebra

A linear algebra A consists of a vector space A over a field $\mathbb{F}$ with an additional vector multiplication $\times$ such that:

1) $v, w \in A \rightarrow v \times w \in A$ closure

2) $\left.\begin{aligned} (v_1 + v_2) \times w &= v_1 \times w + v_2 \times w \\ v \times (w_1 + w_2) &= v \times w_1 + v \times w_2 \end{aligned}\right\}$ bilinearity

2.5. Lie algebra

A Lie algebra g is a linear algebra with the Lie bracket $[\cdot, \cdot]$ as vector product, satisfying:

1) $[x, y] = -[y, x] \quad \forall x, y \in g$

2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ Jacobi identity

Remarks: - if the element of g can be realised as $n \times n$ matrices A, B with

$$[A, B] = A \cdot B - B \cdot A$$

then 1) and 2) hold automatically

- if we do not have matrices:

$$[T_a, T_b] = f_{ab}{}^c T_c$$

structure coefficients

1) $f_{ab}{}^c = -f_{ba}{}^c$

2) $\sum_c \left(f_{ab}{}^c f_{cd}{}^e + f_{bd}{}^c f_{ca}{}^e + f_{da}{}^c f_{cb}{}^e \right) = 0$

2.6. Lie group

A Lie or continuous group is an n-dimensional

manifold G and a mapping $m: G\times G \to G$ such that m defines a group multiplication.
The mappings m and $i: G \to G$, defined as $i(a) = a^{-1}$, are both smooth. Furthermore the map $j(a) = e \;\forall a \in G$ selects the identity element.
inverse element

3. Differential geometry on Lie groups

3.1. Left-invariant tensor fields

We define two maps:

$L_g: G \to G \quad h \to L_g h = g\cdot h$ left translation
$R_g: G \to G \quad h \to R_g h = h\cdot g$ right —"—

or $L_g = m(g, \cdot)$ and $R_g = m(\cdot, g)$

Properties:

i) they are bijective
ii) for Lie groups they are smooth because m is smooth
with i) $\rightsquigarrow$ they are diffeomorphisms of G
iii) the relations

$$L_{gh} = L_g \circ L_h \qquad R_{gh} = R_h \circ R_g$$

$$L_g^{-1} = L_{g^{-1}} \qquad R_g^{-1} = R_{g^{-1}}$$

$$L_g \circ L_h \circ = g\cdot(h\cdot \circ) = (g\cdot h)\cdot \circ = L_{gh} \circ$$

$$L_g^{-1} L_g \circ = L_{g^{-1}} L_g \circ = (g^{-1} g)\cdot \circ = \circ$$

iv) right and left translations commute

$$L_g \circ R_h = R_h \circ L_g$$

$$L_g \circ R_h \circ = g \cdot (\circ \cdot h) = (g \cdot \circ) \cdot h = R_h \circ L_g$$

v) A diffeomorphism f which commutes with all left translations is necessarily the right translation:

$$f \circ L_g = L_g \circ f \quad \Leftrightarrow \quad f(g) = R_{f(e)}(g)$$

A tensor field T of type $\binom{p}{q}$ on G is left-invariant if it satisfies: $\boxed{L_g^* T = T}$

Remember pull back of map $\phi : M \to N$:

1. function $f : N \to \mathbb{R}$, $\phi^* f : M \to \mathbb{R}$

$$(\phi^* f)(x) = f(\phi(x))$$

2. one-form $\alpha \in T^* N$ ← co-tangent space

$\alpha_y : T_y N \to \mathbb{R}$ ← tangent space of N @ point y

$$(\phi^* \alpha)_x (X) = \alpha_{\phi(x)} \left[d\phi_x (X) \right]$$

push forward $d\phi$ or ϕ_*

$$d\phi_x : T_x M \to T_{\phi(x)} N \quad \hat{=} \text{ Jacobi matrix}$$

$$\rightsquigarrow \quad \phi^* \alpha : T^* N \to T^* M$$

Properties:

i) T is uniquely specified by its value at the identity, $T(e)$, because

$$T(g) = L_{g*}\, T(e) \quad \nearrow \boxed{\text{Exercise}}$$

ii) smooth

invariant tensor fields can be constructed from their value @ the identity, only finitely many of them

examples:

1. left-invariant functions $\binom{0}{0}$ = const. functions on G

2. — " — vector fields $\binom{1}{0}$

$$e_a(g) = L_{g*}\, E_a \qquad E_a = e_a(\underset{}{\overset{\text{identity}}{e}})$$

they are dual to

3. — " — one-forms $\binom{0}{1}$

$$e^a(g) = L_{g*}\, E^a \qquad E^a = e^a(e)$$

with pairing $\langle E^a, E_b \rangle = \delta^a_b$

$$\hookrightarrow \langle E^a, E_b \rangle = e^a(g)\big(e_b(g)\big) = \delta^b_a$$

left-invariant scalar