3.2. Energy momentum tensor

Remember: 4 - conserved currents $\underset{\text{energy}}{T_0{}^\mu}$, $\underset{\text{momentum}}{T_i{}^\mu}$

with $T_i{}^0 = \dot\phi\, \partial_i \phi \to P_i = \int dV\, \dot\phi\, \partial_i \phi$

$$\boxed{T_{\mu\nu} := T_\mu{}^\rho \eta_{\rho\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \underbrace{\mathcal{L}}_{1/2(\partial_\rho \phi \partial^\rho \phi - m^2\phi^2)}}$$

(stress) energy (momentum) tensor

- properties: 1) symmetric $T_{\mu\nu} = T_{\nu\mu}$

2) conserved

$$\partial^\mu T_{\mu\nu} = \partial^\mu \partial_\mu \phi \partial_\nu \phi + \cancel{\partial_\mu \phi \partial_\nu \partial^\mu \phi} - \frac{1}{2} 2 \cdot \cancel{\partial_\rho \phi \partial_\nu \partial^\rho \phi} + m^2 \phi \partial_\nu \phi$$

$$= \partial_\nu \phi \left(\partial^\mu \partial_\mu \phi + m^2 \phi \right) = \partial_\nu \phi \underbrace{(-\Box + m^2)\phi}_{\text{e.o.m}}$$

$\leadsto \partial^\mu T_{\mu\nu} = 0$ after imposing the e.o.m.

4 - conservation laws

There are 6 more:

$$u^{[\mu\nu]} := \frac{1}{2}(u^{\mu\nu} - u^{\nu\mu})$$

$$M^{\mu,\nu\rho} = T^{\mu\nu} X^\rho - T^{\mu\rho} X^\nu = 2 T^{\mu[\nu} X^{\rho]}$$

$$\partial_\mu M^{\mu,\nu\rho} = \underbrace{\partial_\mu T^{\mu\nu}}_{=0} X^\rho + T^{\mu\nu} \delta^\rho_\mu - \underbrace{\partial_\mu T^{\mu\rho}}_{=0} X^\nu - T^{\mu\rho} \delta^\nu_\mu$$

$$= T^{\rho\nu} - T^{\nu\rho} = 2 T^{[\rho\nu]} = 0$$

remember $T^{\mu\nu}$ is symmetric!

interpretation: $\xi^\mu = \chi^{\mu\nu} X_\nu - \chi^{\nu\mu} X_\nu$

with $\delta X^\mu = \xi^\mu$ as before

we find: $\quad \partial_\mu \xi^\mu = \chi^{\mu\nu}\eta_{\mu\nu} - \chi^{\mu\nu}\eta_{\mu\nu} = 0$

$$\rightsquigarrow \quad \delta \mathcal{L} = \xi^\mu \partial_\mu \mathcal{L} = \underbrace{\partial_\mu(\xi^\mu \mathcal{L})}_{\text{total derivative}}$$

3.3. Noether's theorem

again we can use $\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \xi^\mu \mathcal{L}}$ from 3.1.

example of Noether's first theorem

and eventually get $J^\mu = \chi_{\nu\varsigma} M^{\mu,\nu\varsigma}$

> "To every continuous symmetry generated by local actions there is a conservent current and vice versa"

symmetry $\hat{=}$ transformation leaving action, e.o.m., ... invariant

4. Special relativity

Question: Origin of KG's $4+6=10$ space-time symmetries?

4.1. Lorentz and Poincaré group

We found transformations of the form $\quad \delta a^\mu = \xi^\mu$, $\quad \delta L^{\mu\nu} = \chi^{[\mu\nu]}$

$$x'^\mu = L^\mu{}_\nu x^\nu + a^\mu \quad (\rightsquigarrow \delta x^\mu = \delta L^\mu{}_\nu x^\nu + \delta a^\mu).$$

They preserve the Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

How? Length element: $\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu}$

$$dx'^\mu = L^\mu{}_\nu\, dx^\nu + \overset{=0}{\overbrace{d(\underbrace{a^\mu}_{=\text{const}})}}$$

$$ds'^2 = ds^2 \rightsquigarrow (\eta'_{\rho\lambda} L^\rho{}_\mu L^\lambda{}_\nu)\, dx^\mu dx^\nu = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

for $\eta'_{\mu\nu} = \eta_{\mu\nu} \rightsquigarrow \boxed{\eta_{\rho\lambda} L^\rho{}_\mu L^\lambda{}_\nu = \eta_{\mu\nu}}$

written as a matrix $L^T \cdot \eta \cdot L = \eta$ $\qquad (L)^{\lambda}{}_{\nu}$ row, column

which matrices are allowed for L?

$$\det(\cancel{\eta}) = \det(L^T \eta L) = \det(L) \cdot \cancel{\det(\eta)} \cdot \det(L)$$

$$1 = \det(L)^2 \rightsquigarrow \det(L) = \pm 1$$

L is non-singular, $\exists\, L^{-1}$ with $L L^{-1} = L^{-1} L = 1$

$$L^T \cdot \eta L \cdot L^{-1} = \eta L^{-1} = L^T \eta \rightarrow L^{-1} = \eta^{-1} L^T \eta$$

or in indices: $\boxed{(L^{-1})^\mu{}_\nu = \eta^{\mu\lambda} L^\rho{}_\lambda \eta_{\rho\nu} = L_\nu{}^\mu}$

More concrete: 1) $L = \mathbb{1}$ ✓ $\qquad \mathbb{1}^T \cdot \eta \cdot \mathbb{1} = \eta$

2) $(L^{-1})^T \cdot \eta \cdot L^{-1} = (\eta^{-1} L^T \eta)^T \cdot \cancel{\eta\, \eta^{-1}} \cdot L^T \eta$

$= \eta^T L \eta^{-T} \cdot L^T \eta$ $\qquad$ with $\eta = \eta^{-1} = \eta^T = \eta^{-T}$

$= \eta L \eta L^T \eta$ $\qquad$ assuming $L^T \eta L = \eta$

$= \eta L L^{-1} = \eta$ //

$\Rightarrow$ if L is allowed, L^{-1} is, too

3) $(L_1 \cdot L_2)^T \cdot \eta \, L_1 \cdot L_2 = L_2^T L_1^T \eta L_1 L_2$

assume $L_{1/2}^T \eta \cdot L_{1/2} = \eta$

$= L_2^T \eta L_2 = \eta$ $\Rightarrow$ if L_1 and L_2 are allowed so is $L_1 \cdot L_2$

1), 2), and 3) are axioms for a group

associativity is automatic for matrices, but we need closure

$\exists$ inverse element

$\exists$ identity element

$\longrightarrow$ Lorentz group $O(3,1)$

Compare to orthogonal group $O(3)$

$ds^2 = \delta_{ij} \, dx^i \, dx^j$ with scalar product

$\langle u^i, v^j \rangle = u^i \delta_{ij} v^j$

norm $\|\vec{u}\| = \langle \vec{u}, \vec{v} \rangle = u^i \, v_i \left(= \sum_{i=1}^{3} u^i u^i \right)$

$\|\vec{u}\| \geq 0$ length of vector

for Lorentz group: $\|\vec{x}\| = x^\mu \eta_{\mu\nu} x^\nu$

take $x^\mu = (t, x, y, z)$ $\|\vec{x}\| = c^2 t^2 - r^2$

$r^2 = x^2 + y^2 + z^2$

with

$\|\vec{x}\| > 0 \qquad c^2 \cdot t^2 > r^2 \quad \hat{=} \quad$ time-like

$\|\vec{x}\| = 0 \qquad c^2 \cdot t^2 = r^2 \quad \hat{=} \quad$ light-like

$\|\vec{x}\| < 0 \qquad c^2 \cdot t^2 < r^2 \quad \hat{=} \quad$ space-like