4. Non-abelian gauge symmetries & Lie groups

so far : reminder of basics from QED

now : push these ideas to learn something new

4.1. Particles & global symmetries

Idea: find larger local symmetries (than $U(1)$)

i.e. two Dirac fermions:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad \text{which transform as}$$

$$\Psi' = \exp\left(i \frac{\alpha^i \sigma_i}{2} \right) \Psi = V\Psi$$

2×2 Pauli matrices mix spinors Ψ_1 and Ψ_2

$\Psi \hat{=}$ column vector with 8 components

$\sigma_i \otimes \mathbb{1}_4$ and $(\sigma_i \Psi)^{\alpha b} = \delta^\alpha_\beta (\sigma_i)^b{}_c \Psi^{\beta c}$

$\sigma_i^\dagger = \sigma_i$ implies $V^\dagger = V^{-1} \leftarrow$ def. unitary matrix $\rightsquigarrow SU(2)$

I. global symmetries:

$$\mathcal{L} = \bar\Psi (i \not\partial) \Psi - m \bar\Psi \Psi \qquad \bar\Psi = \left(\Psi_1^\dagger \gamma_0 \quad \Psi_2^\dagger \gamma_0 \right)$$

$$\bar\Psi' = \Psi'^\dagger \gamma_0 = \Psi^\dagger V^\dagger \gamma_0 = \bar\Psi V^\dagger$$

commute $\quad \gamma_0 = \mathbb{1}_2 \times \gamma_0$

$\Rightarrow \mathcal{L}' = \mathcal{L}$ ✓

II. conserved current:

$$\delta \Psi = \alpha^i \frac{\partial}{\partial \alpha^i} V \Big|_{\alpha^j = 0} \Psi = i \frac{\alpha^i \sigma_i}{2} \Psi \quad \text{(infinitesimal version)}$$

$$\delta S = 0 = \ldots = \int d^4x \, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \Psi)} \delta \Psi(\alpha^i) \right) = \int d^4x \, \partial_\mu J^\mu_i \alpha^i$$

see last lecture, i.e. use field equations

conserved current

$$= \int d^4x\ \partial_\mu \left(-\frac{\alpha^i}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi \right)$$

$$\boxed{J_i^\mu = -\frac{1}{2} \bar{\Psi} \gamma^\mu \sigma_i \Psi} \quad \text{with } \partial_\mu J_i^\mu = 0$$

conserved charges $Q_i = \int d^3x\ J_i^0$

? physical interpretation of Q_i ?

remember canonical momentum $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\Psi}} = i \Psi^\dagger$

$$Q_i = \int d^3x \frac{(i)^2}{2} \Psi^\dagger \sigma_i \Psi = \frac{i}{2} \int d^3x\ \pi \sigma_i \Psi$$

$$\{ Q_i, Q_j \} = -1/4 \int d^3x \int d^3y\ \{ \pi \sigma_i \Psi(\vec{x}), \pi \sigma_j \Psi(\vec{y}) \}$$

↖ Poisson brackets

$$\stackrel{\vdots}{=} -\frac{i}{4} \int d^3x\ \pi [\sigma_i, \sigma_j] \Psi(\vec{x})$$

$$\boxed{[\sigma_i, \sigma_j] = 2i\, \varepsilon_{ij}{}^k \sigma_k} \quad \text{Lie algebra } su(2)$$

$$\hookrightarrow \{ Q_i, Q_j \} = i\, \varepsilon_{ij}{}^k Q_k$$

↓ cannonical quantisation

$$[Q_i, Q_j] = i\, \varepsilon_{ij}{}^k Q_k$$

$\hat{=}$ angular momentum with $Q_i = L_i$

Excurse "Good ol' angular momentum in QM"

we can diagonalise $L^2 = \sum_i L_i^2$ and L_3 at the same time

casimir operator ↗ ↗ cartan generator of $su(2)$

because $[L^2, L_3] = 0$

Hilbert space:

$$L^2 |l\, m\rangle = l(l+1)\, |l\, m\rangle \qquad l = 0, \tfrac{1}{2}, 1, \dots$$

$$L_3 |l\, m\rangle = m\, |l\, m\rangle \qquad m = -l, -l+1, \dots, l$$

spin quantum number

$$|0\,0\rangle \qquad \begin{matrix} |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \\ \downarrow L_- \\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \end{matrix} \qquad \begin{matrix} |1\ 1\rangle \\ \downarrow L_- \\ |1\ 0\rangle \\ \downarrow L_- \\ |1\ -1\rangle \end{matrix} \qquad \vdots$$

singlet, doublet, triplet

analog: eigenvalues of Q_3 = iso spin

$N_u \dots$ number of up quarks $\qquad Q_3 = \frac{1}{2}(N_u - N_d)$

$N_d \dots$ number of down quarks

Doublet = fundamental representation of Lie algebra su(2)

all other irreps from tensor products

i.e. one quark $\qquad |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = u$

$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = d$

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two quarks, spin = 0

$$\left.\begin{array}{ll} |1\ 1\rangle = u\bar{d} & \pi^+ \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) & \pi^0 \\ |1\ -1\rangle = d\bar{u} & \pi^- \end{array}\right\} \text{mesons}$$

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

↗ EX 3.2. for details

→ $\boxed{\text{Particles organise into representations of the global symmetry!}}$

Idea of Yang & Mills: gauge this symmetry

make it local

4.2. The Yang-Mills Lagrangian

we need covariant derivative!

remember $U(1)$: $D_\mu \phi = (\partial_\mu + ie A_\mu)\phi$ connection or gauge field

now: $$\boxed{D_\mu = \partial_\mu - ig A^i_\mu \frac{\sigma_i}{2}}$$

check: $\delta D_\mu \Psi = \partial_\mu \delta\Psi - ig\,\delta A^i_\mu \frac{\sigma_i}{2}\Psi - ig A^i_\mu \frac{\sigma_i}{2}\delta\Psi$

$$= i\alpha^i \frac{\sigma_i}{2} D_\mu \Psi \; \boxed{- \frac{1}{4} g \alpha^i A^j_\mu [\sigma_i, \sigma_j]\Psi + i\partial_\mu \alpha^i \frac{\sigma_i}{2}\Psi - ig\,\delta A^i_\mu \frac{\sigma_i}{2}\Psi} \overset{!}{=} 0$$

$$\leadsto \delta A^i_\mu \frac{\sigma_i}{2} = \frac{1}{g}\partial_\mu \alpha^i \frac{\sigma_i}{2} + \frac{i}{4} g \alpha^i A^j_\mu [\sigma_i, \sigma_j]$$

Field strength: $[D_\mu, D_\nu]\Psi =: -ig F^i_{\mu\nu}\frac{\sigma_i}{2}\Psi$

$$\ldots \quad F^i_{\mu\nu}\frac{\sigma_i}{2} = 2\partial_{[\mu} A^i_{\nu]}\frac{\sigma_i}{2} - ig A^i_\mu A^j_\nu \left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right]$$

simplify by using: $t_i = \frac{\sigma_i}{2}$

$$[t_i, t_j] = i f_{ij}{}^k t_k$$

results in
$$\boxed{\begin{aligned} \delta A^i_\mu &= g^{-1}\partial_\mu \alpha^i - g\alpha^j A^k_\mu f_{jk}{}^i \\ F^i_{\mu\nu} &= 2\partial_{[\mu} A^i_{\nu]} + g A^j_\mu A^k_\nu f_{jk}{}^i \end{aligned}}$$

remark: - holds not just for $SU(2)$ but for any Lie group
- for $f_{ij}{}^k = 0$ results from last lecture

Lagragian: should not transform $\rightarrow$ trivial representation

Killing metric: $\boxed{2\,\mathrm{Tr}(t_i\, t_j) = \mathcal{K}_{ij}}$

compute: $2\,\mathrm{Tr}\left(\frac{\sigma_i}{2}\,\frac{\sigma_j}{2}\right) = \frac{1}{2}\mathrm{Tr}(\sigma_i \sigma_j) = \delta_{ij} = \mathcal{K}_{ij}$

$\mathcal{K}^{ij}$ inverse with $\mathcal{K}^{ik}\mathcal{K}_{kj} = \delta^i_j$

used to raise & lower "Lie alg. indices"

i.e. $F_{i\mu\nu} = \mathcal{K}_{ij} F^j_{\mu\nu}$

the most general Lagragian with two derivatives is

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{D}\psi) - 1/4\, F_{i\mu\nu} F^{i\mu\nu} - c\,\varepsilon^{\mu\nu\sigma\varsigma} F_{i\mu\nu} F^i_{\sigma\varsigma} - m\bar{\psi}\psi}$$

↯ third term violates discret P and T symmetry

parity → P, time reversal → T

$P: x^i \rightarrow -x^i$, $T: x^0 \rightarrow -x^0$

$\leadsto c = 0$ (see QCD Θ-angle)

4.3. The Standard Model

gauge groups realised in nature:

$$SU(3) \times \underbrace{SU(2) \times U(1)}$$ ← broken by Higgs mechanism

quantum chromodynamics (SU(3)) > electroweak interaction

~ strong force

~ electro-magnetic & ~ weak force