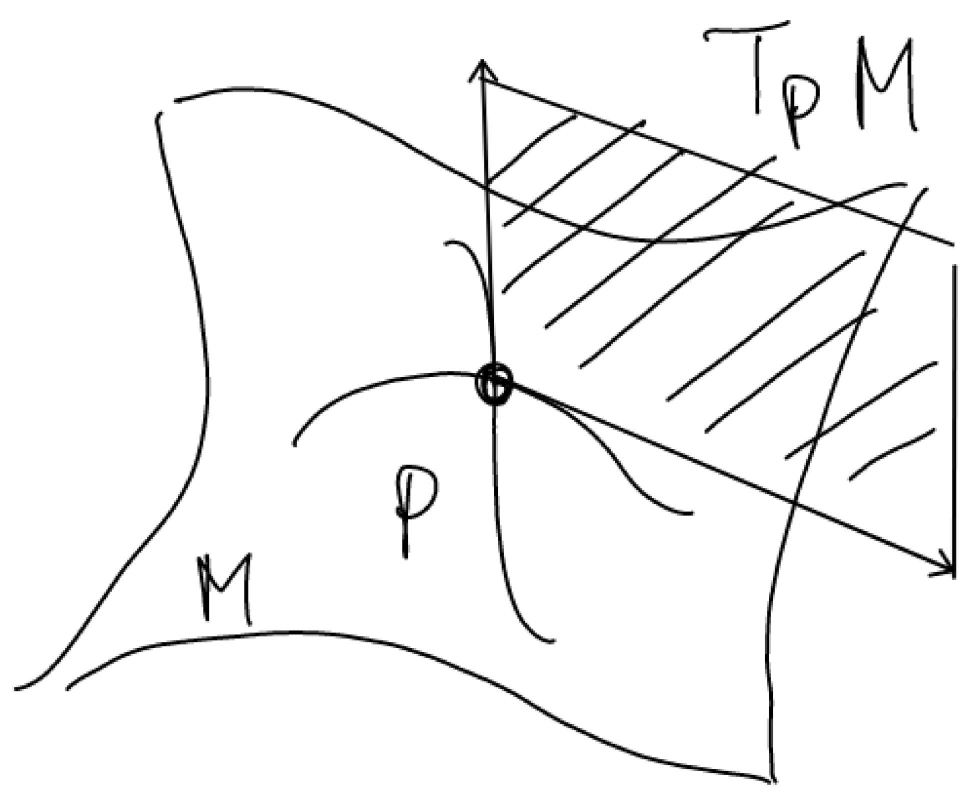
Remember:

last time: Gauge theory (on the boundary) $\xleftrightarrow{\text{duality}}$ Gravity (in the bulk) :today

3 A primer on general relativity

Gravity: Matter tells space how to curve and space tells matter how to move.

3.1 Differential geometry



space = real, d-dim. manifold M with tangent space T_pM on every point p.

= vector space spaned by $\partial_\mu = \frac{\partial}{\partial x^\mu}$ with vectors $\boxed{V = V^\mu(x)\partial_\mu}$

dual vector space = T_p^*M co-tangent space

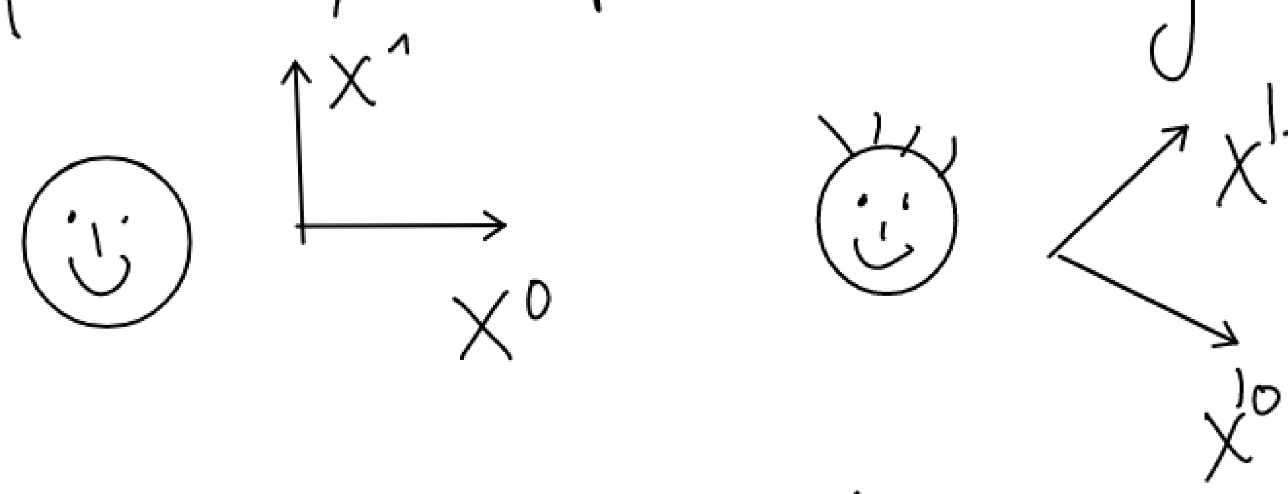
spanded by dx^ν with

$$dx^\nu(\partial_\mu) = i_\mu dx^\nu = \delta_\mu^\nu$$

elements are one-forms, i.e. $\boxed{\varphi = \varphi_\mu(x) dx^\mu}$

3.1.1. Coordinate transformations

Principle of relativity: Laws of physics are the same for any observer.



related by coordinate transformation

like last lecture: $x^\mu \to x'^\mu$ then

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$$

but

$$V = V^\mu \partial_\mu = V'^\mu \partial'_\mu \rightsquigarrow \boxed{V'^\mu = V^\nu \frac{\partial x'^\mu}{\partial x^\nu}}$$

similarly $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu$ and

$$\varphi = \varphi_\mu dx^\mu = \varphi'_\mu dx'^\mu \rightsquigarrow \boxed{\varphi'_\mu = W_\nu \frac{\partial x^\nu}{\partial x'^\mu}}$$

$\Big\}$ tensors

<u>Remarks:</u> • generalizes to

$$\underbrace{T^{(r,s)}}_{\text{rank}} = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s}$$

• may have additional symmetries, like

$$T_{(\nu_1 \dots \nu_n)\nu_{n+1} \dots \nu_s} = \frac{1}{n!}\left(T_{\nu_1 \dots \nu_n \nu_{n+1} \dots \nu_s} + \text{perm. of } \nu_1 \dots \nu_n \right)$$

$$T_{[\nu_1 \dots \nu_n]\nu_{n+1} \dots \nu_s} = \frac{1}{n!}\left(\text{-"-} \pm \text{alternating -"-} \right)$$

3.1.2 Metric and vielbeins

symmetric, non-degenerate $(0,2)$ tensor $g_{\mu\nu}$

$g_{\mu\nu} \overset{?}{=} g_{\nu\mu}$? $\det(g_{\mu\nu}) \neq 0 \longrightarrow \exists\, g^{\mu\nu}$ with $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$

compact way of writting it $\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$

<u>Examples:</u> • Eucledian space $ds^2 = (dx^1)^2 + (dx^2) + (dx^3)^2$

• 2-sphere: $x^1 = R\sin\phi$, $x^2 = R\cos\phi\sin\vartheta$, $x^3 = R\cos\phi\cos\vartheta$

$$dx^1 = R\cos\phi\, d\phi$$
$$dx^2 = -R\sin\phi\sin\vartheta\, d\phi + R\cos\phi\cos\vartheta\, d\vartheta$$
$$dx^3 = -R\sin\phi\cos\vartheta\, d\phi - R\cos\phi\sin\vartheta\, d\vartheta$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = R^2(d\phi^2 + \cos^2\phi\, d\vartheta^2)$$

Remarks: • used to raise/lower indices

$$V_\mu = g_{\mu\nu} V^\nu \quad \text{or} \quad \varphi^\mu = g^{\mu\nu} \varphi_\nu$$

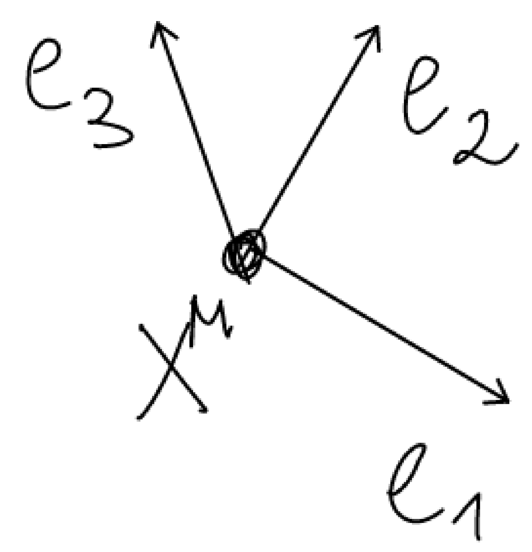
• always writeable as $\boxed{g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu}$

const, and diag, either Lorentzian or Eucledian

spans $T^* M$

vielbein $\hat{=}$ many leg(s) $e_a{}^\mu$, with $e_a{}^\mu e^b{}_\mu = \delta_a^b$

spanning TM



Change of basis $e'_a(x) = \Lambda_a{}^b(x)\, e_b(x)$

with $\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab}$ called local Lorentz transf.

• tensors with flat (Latin) and curved (Greek) indices

like $T_a{}^\mu = e_a{}^\nu T_\nu{}^\mu$ transform as

$$T'_a{}^\mu = \Lambda_a{}^b T_b{}^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

3.1.3 Covariant derivative

Remember YM-theory, ∂_μ transformed not „nicely" ($\hat{=}$ covariantly)

$$\longrightarrow \quad D_\mu = \partial_\mu + i[A_\mu, \;]$$

same here:

$$\partial'_\mu \varphi'_\nu = \frac{\partial}{\partial x'^\mu} \varphi'_\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x^\sigma}{\partial x'^\nu} \varphi_\sigma \right)$$

$$= \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \left(\frac{\partial}{\partial x^\rho} \varphi_\sigma \right)}_{\text{covariant}} + \underbrace{\varphi_\sigma \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x^\sigma}{\partial x^\rho \partial x'^\nu}}_{\substack{\text{anomalous} \\ \text{has to be compensated}}}$$

$$\Rightarrow \boxed{\begin{aligned} \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \\ \nabla_\mu \varphi_\nu &= \partial_\mu \varphi_\nu - \Gamma^\lambda_{\mu\nu} \varphi_\lambda \end{aligned}}$$

Christoffel symbols or Levi-Civita connection

Properties: • $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ (symmetric, or torsion-less)

• $\nabla_\mu g_{\nu\rho} = 0$ (metric compatible)

fixes $\Gamma^\lambda_{\mu\nu}$ completely in terms of $g_{\mu\nu}, g^{\mu\nu}, \partial_\mu$

↗ EX 3.2

• $\nabla_\mu (T+S) = \nabla_\mu T + \nabla_\mu S$ (linear)

• $\nabla_\mu (S\cdot T) = (\nabla_\mu S)\cdot T + S\cdot(\nabla_\mu T)$ (Leibniz)

For mixed tensors like $e^a{}_\nu$, we have

$$\nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\lambda_{\mu\nu} e^a{}_\lambda$$

spin connection is fixed by

requiring $\boxed{\nabla_\mu e^a{}_\nu = 0} \Rightarrow \omega_\mu{}^a{}_b = e^a{}_\lambda e_b{}^\nu \Gamma^\lambda_{\mu\nu} - e_b{}^\nu \partial_\mu e^a{}_\nu$

3.1.4 Lie derivative

describes the infinitesimal version of coordinate transform.

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$$ infinitesimal

$$V'^\mu = V^\mu + \mathcal{L}_\xi V^\mu$$ with

$$\mathcal{L}_\xi V^\mu = \xi^\rho \partial_\rho V^\mu - V^\rho \partial_\rho \xi^\mu$$ or

$$\boxed{\mathcal{L}_\xi T^\mu{}_\nu = \xi^\rho \partial_\rho T^\mu{}_\nu - (\partial_\rho \xi^\mu) T^\rho{}_\nu + (\partial_\nu \xi^\rho) T^\mu{}_\rho}$$

again, it is linear $\mathcal{L}_\xi (T+S) = \mathcal{L}_\xi T + \mathcal{L}_\xi S$ and satisfies the Leibnitz rule $\mathcal{L}_\xi (T \cdot S) = \mathcal{L}_\xi T \cdot S + T \cdot \mathcal{L}_\xi S$

3.1.5 Differential forms

A particular useful subset of $(0,p)$ tensors are the antisymmetric p-forms $\omega_{\mu_1 \dots \mu_p} = \omega_{[\mu_1 \dots \mu_p]}$

which we write as $\boxed{\omega^{(p)} = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}}$

p-form

one-forms

form degree	example
0	$f(x)$
1	$df(x)$
2	$df(x) \wedge dg(x)$

exterior derivative

$$df(x) = \partial_\mu f \, dx^\mu$$

wedge Product

$$\boxed{\begin{aligned} \omega^{(p)} \wedge \varphi^{(q)} &= (-1)^{pq} \varphi^{(q)} \wedge \omega^{(p)} \\ d(\omega^{(p)} \wedge \varphi^{(q)}) &= d\omega^{(p)} \wedge \varphi^{(q)} + (-1)^p \omega^{(p)} \wedge d\varphi^{(q)} \\ d^2 \omega^{(p)} &= 0 \end{aligned}}$$