1.2.2. Outlook representations

Can we find other (real) matrices E_i' with the same commutators?

Yes! i.e. product representation

$$E_i' = E_i \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes E_i$$

Kronecker product with

$\hat{=}$ coupled angular momentum in QM $so(3) = su(2)$

$$A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & & \vdots \\ a_{n1} B & \cdots & a_{nn} B \end{pmatrix} \qquad \begin{matrix} A \in M_{n \times n} \\ B \in M_{m \times m} \end{matrix}$$

$$A \otimes B \in M_{n \cdot m \times n \cdot m} \rightarrow E_i' \in M_{9 \times 9 = 3 \cdot 3}$$

we can check: $[E_i', E_j'] = [E_i, E_j] \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes [E_i, E_j]$
$$= \sum_{k=1}^{3} \varepsilon_{ijk} E_k'$$

BUT not an irrep! $\rightarrow$ use Casimir operator

$$C := -E_1'^2 - E_2'^2 - E_3'^2 \;\hat{=}\; \vec{L}^2 \text{ in QM}$$

1. find basis in which C is diagonal
2. transform E_i' into this basis

$$C' = S^{-1} C S = \text{diag}(\underbrace{6, \ldots, 6}_{5\times}, \underbrace{2, \ldots, 2}_{3\times}, 0)$$

$$E_i'' = S^{-1} E_i' S = \begin{pmatrix} \boxed{5\times5} & 0 & 0 \\ 0 & \boxed{3\times3} & 0 \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

irreps

We say: $3 \times 3 \rightarrow 1 + 3 + 5$

2. Mathematical fundations

this course: ideas from $SO(3)$ $\xrightarrow[\text{generalise}]{}$ much more complicated Lie groups & algebras

But first define relevant objects!

2.1. Group

Def: A group is a set G with an operation, called multiplication, $\circ$ such that:

1) $g_1, g_2 \in G \longrightarrow g_1 \circ g_2 \in G$ closure
2) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$, $g_1, g_2, g_3 \in G$ associativity
3) $\exists\, e \in G$ such that $g = e \circ g = g \circ e$ $\forall\, g \in G$ exisitence of identity
4) $\forall\, g \in G, \exists\, g^{-1}$ such that $g \circ g^{-1} = g^{-1} \circ g = e$ existence of inverse

If in addition

5) $g \circ g' = g' \circ g$ $\forall\, g, g' \in G$ commutative

holds, the group is called abelian.

Examples:
- permutation group S_n of n ordered elements has $n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$ group elements $\rightarrow$ finite group
- $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ $a, b \in \mathbb{Z}_n$

  $a \circ b = (a + b) \bmod n$

  identity: $e = 0$

  inverse: $(a^{-1} + a) \bmod n = e = 0$

$a^{-1} = (-a) \bmod n$

$\lightning$ a^{-1} = group inverse <u>not</u> $a^{-1} = \frac{1}{a}$

check: $a \in \mathbb{Z}_n \rightarrow a^{-1} \in \mathbb{Z}_n$ ✓

Note: A smallest set of $\{g_1, \ldots, g_N\} \in G$ such that any element $g \in G$ can be obtained as product of them is called a basis of G. Its elements are called generators.

2.2. Field

A field $\mathbb{F}$ is a set with two operations: addition $+$ and scalar multiplication $\circ$ such that:

1) $\mathbb{F}$ is an abelian group under $+$ with 0 as identity element
2) $\mathbb{F} - \{0\}$ is a group under multiplication with 1 as identity
3) $a, b, c \in \mathbb{F} \Rightarrow \begin{cases} a \cdot (b+c) = a \cdot b + a \cdot c \\ (a+b) \cdot c = a \cdot c + b \cdot c \end{cases}$
   distributivity

If additionally:

4) $a \cdot b = b \cdot a$ the field is called commutative

relevant in physics:

- real numbers $\mathbb{R} \ni a, b$
- complex numbers $\mathbb{C} \ni c = a + ib$, $i^2 = -1$, $i = \sqrt{-1}$
- quaternions $\mathbb{H}$

$i^2 = j^2 = k^2 = -1$ $\qquad$ $i \cdot j = -j \cdot i = k$

$j \cdot k = -k \cdot j = i$

$k \cdot i = -i \cdot k = j$

$\hookrightarrow \mathbb{H} \ni q = a + ib + jc + kd \qquad a, b, c, d \in \mathbb{R}$

$= (a + ib) + (c + id) j$

4 real = 2 complex parameters

$\hookrightarrow$ quaternions are non-commutative

Complex conjugation

$\mathbb{C}: \quad (1, i)^* = (1^*, i^*) = (1, -i)$

$\mathbb{H}: \quad (1, i, j, k)^* = (1^*, i^*, j^*, k^*) = (1, -i, -j, -k)$

sometimes we also use $\bar{\ }$ instead of $*$

2.3. Vector space

A vector space V over a field $\mathbb{F}$ satisfies:

1) V is an abelian group under addition $+$
2) $a \in \mathbb{F}, \quad v \in V \Rightarrow \quad a \cdot v \in V$ $\qquad$ closure

$\qquad$ scalar multiplication

3) $a \cdot (v + w) = a \cdot v + a \cdot w \qquad v, w \in V$ $\qquad$ bilinearity

$(a + b) \cdot w = a \cdot w + b \cdot w \qquad a, b \in \mathbb{F}$

- Elements $v \in V$ are called vectors

- A basis B for V is a minimal set of N vectors $e_i \in V$ such that any vector $v \in V$ can be represented as

$v = \sum_i a_i e_i$ with $a_i \in \mathbb{F}$

- The number N of basis vectors is called dimension of V: $\dim_{\mathbb{F}}(V) = N$ ← depends on the field we use

in particular we have
$$4 \dim_{\mathbb{H}}(V) = 2 \dim_{\mathbb{C}}(V) = \dim_{\mathbb{R}}(V)$$

Combining vector spaces

(I.) direct sum $V_1 \oplus V_2$, both over same field $\mathbb{F}$

defined by:

1) $a \cdot (v_1 \oplus v_2) = (a v_1) \oplus (a v_2) \qquad a \in \mathbb{H}$
2) $v_1 \oplus v_2 + w_1 \oplus w_2 = (v_1 + w_1) \oplus (v_2 + w_2) \qquad v_1, w_1 \in V_1,\ v_2, w_2 \in V_2$

in practice: $v = v_1 \oplus v_2 = \begin{pmatrix} v_{1i} \\ v_{2j} \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} v_{11} \\ \vdots \\ v_{1n} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} v_{21} \\ \vdots \\ v_{2m} \end{pmatrix}$

(II.) cartesian product $V_1 \otimes V_2$, again over the same field $\mathbb{F}$,

is the set of tuples (v_1, v_2) which satisfy:

1) $(a v_1, v_2) = (v_1, a v_2) \qquad a \in \mathbb{F}$
2) $(v_1 + w_1, v_2 + w_2) = (v_1, v_2) + (v_1, w_2) + (w_1, v_2) + (w_1, w_2)$

or in practice: $v_{ij} = v_{1i} \cdot v_{2j}$

For the dimensions
$$\dim_{\mathbb{F}}(V_1 \oplus V_2) = \dim_{\mathbb{F}}(V_1) + \dim_{\mathbb{F}}(V_2)$$
$$\dim_{\mathbb{F}}(V_1 \otimes V_2) = \dim_{\mathbb{F}}(V_1) \cdot \dim_{\mathbb{F}}(V_2)$$
holds.

Remember $\quad 3 \otimes 3 = 1 \oplus 3 \oplus 5 \quad$ from the beginning