1.2. Klein-Gordon field

a) structure of sine-Gordon

for small Φ $\sin\Phi \simeq \Phi$

$$-\frac{\partial^2\Phi}{\partial\tau^2} + \frac{\partial^2\Phi}{\partial\xi^2} - \Phi = 0$$

2^{nd} deriv. wrt time – wrt space

We generalize to

$$\boxed{\Box\varphi - \mu^2\varphi = 0}$$ with $\mu = \frac{m\cdot c}{\hbar}$, and

$$\Box\varphi = -\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Units:
c = speed of light
m_e = mass of electron
$\hbar$ = reduced Plank constant

$$\left.\begin{aligned} \text{length} &= \hbar/(m_e c) \\ \text{mass} &= m_e \\ \text{time} &= \hbar/(m_e c^2) \end{aligned}\right\}$$

check velocity $= \frac{\text{length}}{\text{time}} = c$

$$[\mu] = \frac{m_e \cdot c}{\hbar} = \frac{1}{\text{length}}$$

$$[\Box] = 1/\text{length}^2$$

Easier in natural units $c = \hbar (= \varepsilon_0) = 1$:

$$\Box\varphi - m^2\varphi = 0$$

linear PDE with constant (m^2) coefficient :-)

⇝ solve by Fourier transformation

$$\varphi(t,\vec{r}) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} d^3k \ \hat{\varphi}_{\vec{k}}(t) \ e^{i\vec{k}\cdot\vec{r}}$$

with "momentum" $\vec{k} = (k_x, k_y, k_z) \in \mathbb{R}^3$

$\varphi(t, \vec{r})$ is real $(\varphi(t,\vec{r})^* = \varphi(t,\vec{r}))$, therefore

$$\hat{\varphi}_{-\vec{k}}(t) = \hat{\varphi}(t)^{*}_{\vec{k}} \quad \text{complex conjugation} \qquad (3)$$

into KG equation

$$\ddot{\hat{\varphi}}_{\vec{k}}(t) + \underbrace{(\vec{k}^2 + m^2)}_{\omega_{\vec{k}}^2} \hat{\varphi}_{\vec{k}}(t) = 0 \quad \text{harmonic oscillator :-)}$$

ansatz: $\hat{\varphi}_{\vec{k}}(t) = a_{\vec{k}}\, e^{-i\omega_{\vec{k}} t} + b_{\vec{k}}\, e^{i\omega_{\vec{k}} t}$

with $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$,

and $b_{-\vec{k}} = a^*_{\vec{k}}$ because of (3)

full solution:

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} d^3k \left(a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r} - i\omega_{\vec{k}} t} + a^*_{\vec{k}}\, e^{-i\vec{k}\vec{r} + i\omega_{\vec{k}} t} \right)$$

$\Rightarrow$ KG field is just an ∞-number of uncoupled harmonic oscillators with frequency $\omega_{\vec{k}}$.

2. Variational principle

2.1. Reminder on classical mechanics

equations of motion $\xleftarrow{\text{Euler-Lagrange eq.}}$ action S

1) $\mathcal{L}(\underbrace{q_n(t), \dot{q}_n(t)}_{\text{configuration (space)}}, t) := T - U$ Lagrangian

kinetic energy $\swarrow \quad \searrow$ potential energy

example non-interacting particles $T = \frac{1}{2}\sum_{n=1}^{N} m_n \dot{q}_n^2$, $V = 0$

2) $S = \int dt\, \mathcal{L}(q_n(t), \dot{q}_n(t), t)$ action

3) $\delta S = 0 \longrightarrow$ EoM Variational principle

$$\delta S = \sum_{n=1}^{N} \int dt \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n} \frac{d}{dt} \delta q_n(t) + \frac{\partial \mathcal{L}}{\partial q_n} \delta q_n(t) \right)$$

integration by parts: $\int_a^b dt \frac{d}{dt}\left(f(x)\right) = f(b) - f(a)$

$$\int_a^b dt\, u(t)\, \dot{v}(t) = -\int_a^b dt\, \dot{u}(t)\, v(t)$$

$= 0$

boundary conditions: $\delta q_n(\pm\infty) = 0$

$\rightsquigarrow$ q_n's are fixed at the beginning and in the end

$$\delta S = \sum_{n=1}^{N} \int dt \left(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} + \frac{\partial \mathcal{L}}{\partial q_n} \right) \delta q_n(t) = 0$$

has to hold for arbitrary functions $\delta q_n(t)$

$$\rightsquigarrow \boxed{-\frac{d}{dt}\frac{\mathcal{L}}{\delta \dot{q}_n} + \frac{\delta \mathcal{L}}{\delta q_n} = 0}$$ Euler-Largange equation

for our example $\mathcal{L} = \frac{1}{2}\sum_{n=1}^{N} m_n \dot{q}_n^2$, $\frac{\delta \mathcal{L}}{\delta q_n} = 0$,

$$\leadsto \quad \underline{\underline{m_n \ddot{q}_n = 0}} \qquad\qquad \frac{\delta L}{\delta \dot{q}_n} = m_n \dot{q}_n$$

2.2. Field theory

now the Lagrangian depends on space and time

$\mathcal{L} = \mathcal{L}(\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{x}), t)$ and we integrate in the action accordingly

$$S = \int dt \int_R d^3x \; \mathcal{L}$$

relevant spacial region

for the KG field, we then get

$$S = \int dt \int_R d^3x \; \frac{1}{2}\left(\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right)$$

equations of motion still follow from $\delta S = 0$