3. The Polyakov action

Remember: last lecture's Nambu-Goto action

$$S_{NG} = -T \int_{\tau_i}^{\tau_f} \int_0^{\sigma'} d\sigma \sqrt{(\dot{X}X')^2 - \dot{X}^2 X'^2}$$

↯ $\sqrt{}$ is a problem for quantisation

→ Same trick as for the point particle, use "aux." metric

→ Polyakov action

Gauge symmetry for reparametrisation ⇝ fix it!

Observe that $\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$, the induced metric, can be broght into conformal gauge into the form

$$\underbrace{\gamma_{\alpha\beta} = \sqrt{-\gamma}}_{\text{scaling factor, preserves angles of geodesics} \rightarrow \text{conformal}} \; \underbrace{\eta_{\alpha\beta}}_{\text{Minkowski metric in two dimensions}}$$

Why?

$$ds^2 = \gamma_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta \qquad \sigma'^\alpha = \sigma'^\alpha(\sigma)$$

$$= \gamma'_{\alpha\beta}\, d\sigma'^\alpha d\sigma'^\beta \qquad d\sigma'^\alpha = \frac{\partial \sigma'^\alpha}{\partial \sigma^\beta} d\sigma^\beta$$

$$\rightsquigarrow \gamma'_{\alpha\beta} = \frac{\partial \sigma^\gamma}{\partial \sigma'^\alpha} \frac{\partial \sigma^\delta}{\partial \sigma'^\beta} \gamma_{\gamma\delta} \qquad S_\alpha{}^\beta = \frac{\partial \sigma^\beta}{\partial \sigma'^\alpha}$$

$$= (S \cdot \gamma \cdot S^T)_{\alpha\beta}$$

and why conformal gauge? e.o.m. for N-G

$$\delta S_{NG} = \delta\left(-T \int d^2\sigma \sqrt{-\gamma}\right) = \frac{T}{2} \int d^2\sigma \sqrt{-\gamma}\, \gamma_{\alpha\beta}\, \delta\gamma^{\alpha\beta} = 0$$

$$= -T \int d^2\sigma \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X_\mu\, \partial_\beta \delta X^\mu$$

$$\Rightarrow \partial_\alpha \left(\sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\beta X_\mu\right) = 0$$

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta X_\mu = 0 \quad \text{Wave equation in 2D}$$

Which i.e. originates from the action

$$S = \frac{1}{2}\int d^2\sigma \; \eta^{\alpha\beta}\, \partial_\alpha X^\mu \,\partial_\beta X_\mu$$

with the gauge fixing constraint $\partial_\alpha X^\mu \partial_\beta X_\mu = \sqrt{-\gamma}\,\eta_{\alpha\beta}$

or $\partial_\alpha X\, \partial_\beta X - 1/2\, \eta_{\alpha\beta}\, \eta^{\gamma\delta}\, \partial_\gamma X\, \partial_\delta X = 0 = \eta_{\alpha\gamma} T^\gamma{}_\beta$

remember: energy momentum tensor

$$\boxed{T^\alpha{}_\beta = \frac{\partial \mathcal{L}}{\partial\, \partial_\alpha X^\mu}\, \partial_\beta X^\mu - \delta^\alpha{}_\beta \mathcal{L}} \quad \text{for } \mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha X \partial_\beta X$$

$$= \partial^\alpha X\, \partial_\beta X - 1/2\, \delta^\alpha{}_\beta\, \partial_\gamma X\, \partial^\gamma X$$

↳ gauge fixing implies $T^\alpha{}_\beta = 0$

The are exactly the field equations of the Polyakov action

$$\boxed{S_P[X, h] = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\; h^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu}$$

with $T_{\alpha\beta} = -\frac{2}{T\sqrt{-h}}\frac{\delta S_P}{\delta h^{\alpha\beta}}$

Conclusion: classical string dynamics is governed by

1. linear 2^{nd} order equation

   $\eta^{\alpha\beta}\,\partial_\alpha \partial_\beta X^\mu = 0$

2. and two constraints: $T_{\alpha\beta}$ is symmetric

   $\rightsquigarrow \frac{2\cdot 3}{2} = 3$ independent components

   but $T_\alpha{}^\alpha = 0$

   can be written as $(\mathring{X} \pm X')^2 = 0$

3.1. Local rescalings of metric

↯ e.o.m + gauge condition (① and ②) do not fix $h_{\alpha\beta}$ completely

reason: Polyakov action has an additional local symmetry, Weyl invariance
= local rescalings of the metric $h_{\alpha\beta}$

note Weyl invariance $\Leftrightarrow$ $T_\alpha{}^\alpha = 0$ (later much more)

Polyakov's insight was to elevate Weyl invariance to a fundamental principle in ST

i.e. $\int d^2\sigma \sqrt{-h}\,\Lambda$ ($\hat{=}$ cosmological constant) is not allowed because it would break Weyl invariance.

$\rightsquigarrow$ massless strings only ?

4. Symmetries of the classical string

4.1. Global symmetries

Remember: Noether's theorem $\to$ for every global continuous symmetry there is a conserved charge

a) symmetry: transformation on the fields, that leaves action invariant

b) global: transformation is the same everywhere

assume we have $X^\mu(\sigma) \to X^\mu(\sigma) + V^\mu(X)$ ← small

$$\delta_V S = 0 = \int d^2\sigma \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha X)} \partial_\alpha V + \frac{\delta \mathcal{L}}{\delta X} V \right)$$

Integration by Parts:

$$0 = \int d\sigma^2 \left[\partial_\alpha \left(\frac{\delta \mathcal{L}}{\delta(\partial_\alpha X)} V \right) - \overbrace{\left(\partial_\alpha \frac{\delta \mathcal{L}}{\delta(\partial_\alpha X)} - \frac{\delta \mathcal{L}}{\delta X} \right)}^{\text{Euler - Lagrange equations} = 0} V \right]$$

$$\rightsquigarrow \boxed{\partial_\alpha j^\alpha = 0 \quad \text{with} \quad j^\alpha = \frac{\delta \mathcal{L}}{\delta(\partial_\alpha X)} V}$$

conserved current (under e.o.m.)

Fix time coordinate τ $(= \sigma^0)$ and space coordinate σ $(= \sigma^1)$ on world sheet

$$0 = \int d^2\sigma \, \partial_\alpha j^\alpha = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma'} d\sigma \left(-\partial_\tau j^\tau + \partial_\sigma j^\sigma\right)$$

$$= -\int_0^{\sigma'} d\sigma \, j^\tau \Big|_{\tau_i}^{\tau_f} + \underbrace{\int_{\tau_i}^{\tau_f} d\tau \, j^\sigma \Big|_0^{\sigma'}}_{=0 \text{ for periodic boundary conditions (closed string)}} = 0$$

$$\rightsquigarrow Q(\tau_i) = Q(\tau_f) \quad \text{with} \quad \boxed{Q = \int_0^{\sigma'} d\sigma \, j^\tau}$$

c) conserved charge (under e.o.m)

We can now introduce the canonical momentum

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial (\underbrace{\partial_\tau X^\mu}_{= \dot{X}^\mu})}$$ and the Hamiltonian

① $$H(\tau) = \int_0^{\sigma'} d\sigma \left[\Pi \cdot \dot{X} - \mathcal{L} \right]$$

To obtain from it the e.o.m. we further need

② equal time Poisson brackets

$$\{ X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma') \} = \delta^\mu_\nu \, \delta(\sigma - \sigma')$$

③ for all functions $f(X, \Pi)$ we have

$$\boxed{\frac{\partial}{\partial \tau} f = \dot{f} = \{ f, H \}}$$

for $f = X$ and Π this gives the e.o.m.

Therefore we find that $Q_V = \int_0^{\sigma'} d\sigma \, \Pi \cdot V$ generate the global symmetry by the action

$\delta_V f = \{Q, f\}$ (and in particular $\{Q, H\} = 0$)

Question: What happens if we apply two such transformation after each other?

$$\underbrace{\delta_{V_1} \delta_{V_2} f}_{\{Q_1, \{Q_2, f\}\}} - \underbrace{\delta_{V_2} \delta_{V_1} f}_{\{Q_2, \{Q_1, f\}\}} = \{\{Q_{V_1}, Q_{V_2}\}, f\} = \delta_{V_{12}} f$$

Leibniz rule for $\{\cdot,\cdot\}$

with $Q_{V_{12}} = \{Q_{V_1}, Q_{V_2}\}$

$\rightsquigarrow$ Q_V's generate a Lie algebra

Example: Poincaré charges of string in conformal gauge.

remember from last lecture: $X^\mu \to X^{\mu'} = \Lambda^\mu{}_\nu X^\nu + a^\mu$

2) Lorentz rotations 1) translations

1) $V_\mu{}^\nu = \delta_\mu{}^\nu$ $P_\mu = Q_{V_\mu} = -T \int_0^{2\pi = \sigma'} d\sigma \, \dot{X}_\mu$

2) ... $J_{\mu\nu} = \int_0^{2\pi} d\sigma \left(\Pi_\mu X_\nu - \Pi_\nu X_\mu \right)$

with $\Pi_\mu = -T \dot{X}_\mu$

EX: Verify that the conserved charges generate the Poincaré algebra.