3. The electromagnetic (spin 1) field

3.1. Complex Klein-Gordon field

Idea: 2 real scalar fields ϕ_1, ϕ_2 ⇝ 1 complex scalar ϕ

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\bar{\phi} = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \qquad \bar{\phi}_{1/2} = \phi_{1/2} \text{ (real)}$$

$$\mathcal{L}(\phi) = \mathcal{L}_{KG}(\phi_1) + \mathcal{L}_{KG}(\phi_2)$$

remember

$$\mathcal{L}_{KG}(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2$$

$$\boxed{\mathcal{L}(\phi) = \partial_\mu\phi\,\partial^\mu\bar{\phi} - m^2\phi\bar{\phi}}$$

Field equations: ↯ treat ϕ and $\bar{\phi}$ as independent

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \Rightarrow \partial_\mu\partial^\mu\bar{\phi} + m^2\bar{\phi} = 0$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\phi}} = 0 \Rightarrow \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

Note: Lagrangian $\mathcal{L}$ is invariant under

$$\phi \to e^{-i\Lambda}\phi \quad \text{and} \quad \bar{\phi} \to e^{i\Lambda}\bar{\phi}$$

Λ is a real constant. ~ Lie group $U(1)$ ↓ Lie algebra $u(1)$

3.2. Conserved currents and charges

infinitesimal: $\delta\phi = -i\phi\Lambda$,

$\delta\bar{\phi} = i\bar{\phi}\Lambda$

$$\delta S = 0 = \int d^4x\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \ldots\delta\bar{\phi}\ldots\right]$$

Integration by parts

$$\overset{\Delta}{=} \int d^4x\ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi(\Lambda) + \ldots\delta\bar{\phi}(\Lambda)\ldots\right) -$$

$$\int d^4x \left[\left(\underbrace{\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi}}_{\text{field equations for } \bar{\Phi} = 0 \text{ (on-shell)}} \right) \delta \phi(\Lambda) + \dots \delta \bar{\Phi}(\Lambda) \right] = 0$$

therefore $\delta S = 0 = \int d^4x \, \partial_\mu J^\mu$ with

$$\boxed{J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta \phi(\Lambda) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\Phi})} \delta \bar{\Phi}(\Lambda)}$$

here $J^\mu = i\left(\bar{\Phi} \partial^\mu \phi - \phi \partial^\mu \bar{\Phi}\right)$

J^μ = conserved current with $\partial_\mu J^\mu = 0$ under field eq.

$$0 = \int_{t_1}^{t_2} dx^0 \left(\partial_0 \underbrace{\int d^3x \, J^0}_{Q(x^0) = Q(t)} - \underbrace{\int d^3x \, \partial_i J^i}_{= 0} \right)$$ because fields vanish @ ∞

$0 = Q(t_2) - Q(t_1) \quad \rightarrow \quad Q(t_2) = Q(t_1)$ or
$Q(t) = \text{const}$

$\boxed{Q(t) = \int d^3x \, J^0}$ conserved charge

Noether's theorem: symmetry $\Longleftrightarrow$ conserved charges

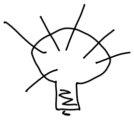
3.3. Gauge symmetries

- parameter Λ does not depend on X^μ !

→ global symmetry (everywhere the same)

Question: Can we make it local, i.e. $\Lambda(X^\mu)$?

But then $\delta \mathcal{L} = \dots = (\partial_\mu \Lambda) J^\mu \neq 0$


gauge $\mathcal{L}$ (compensate with additional terms)

$$\mathcal{L}_1 = - e J^\mu A_\mu$$ ← new field called gauge field with

$$\boxed{\delta A_\mu = \frac{1}{e}\partial_\mu \Lambda}$$

$$\delta \mathcal{L}_1 = - e\, \delta J^\mu A_\mu - J^\mu \partial_\mu \Lambda$$ ← term that we want

↑ but we also get this one :-c

$$\delta J^\mu = 2 \bar{\phi} \phi \partial^\mu \Lambda \quad \rightarrow \quad \delta(\mathcal{L} + \mathcal{L}_1) = -2 e A_\mu \partial^\mu \Lambda \phi \bar{\phi}$$

<u>compensate again!</u>

$$\mathcal{L}_2 = e^2 A_\mu A^\mu \phi \bar{\phi} \quad \rightarrow \quad \delta \mathcal{L}_2 = 2 e A_\mu \partial^\mu \Lambda \phi \bar{\phi}$$:-)

$$\begin{aligned} \mathcal{L}_{\text{gauged}} &= \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 \\ &= (\partial_\mu \phi + i e A_\mu \phi)(\partial^\mu \bar{\phi} - i e A^\mu \bar{\phi}) - m^2 \phi \bar{\phi} \; // \end{aligned}$$

↯ Symmetry of this Lagrangian is hard to see.
Can we do better?

<u>Covariant derivatives:</u>

$$\boxed{D_\mu \phi = (\partial_\mu + i e A_\mu)\phi}$$ with

$$\begin{aligned} \delta(D_\mu \phi) &= \delta(\partial_\mu \phi) + i e \,\delta A_\mu \phi + i e A_\mu \delta \phi \\ &= -i \partial_\mu (\Lambda \phi) + i \cancel{e} \frac{1}{\cancel{e}} \partial_\mu \Lambda \phi + e A_\mu \Lambda \phi \\ &= -i \Lambda (\partial_\mu \phi + i e A_\mu \phi) = -i \Lambda (D_\mu \phi) \; // \end{aligned}$$

$D_\mu \phi$ transforms like ϕ, namely <u>covariantly</u>

$$\boxed{\mathcal{L}_{\text{gauged}} = D_\mu \phi \overline{D^\mu \phi} - m^2 \phi \phi}$$

Question: Can we generate more covariant quantities?

Yes! i.e.

$$[D_\mu, D_\nu]\phi = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\phi - (\mu \leftrightarrow \nu)$$

$$= \cancel{\partial_\mu \partial_\nu \phi} + ie\partial_\mu(A_\nu \phi) + \cancel{ieA_\mu \partial_\nu} \phi - ie^2 A_\mu A_\nu \phi$$
$$- (\mu \leftrightarrow \nu)$$

$$= ie \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} \phi$$

$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$ = electromagnetic field tensor

$\delta F_{\mu\nu} = 2 \frac{1}{e} \partial_{[\mu} \partial_{\nu]} \Lambda = 0$ // $X_{[\mu\nu]} = 1/2(X_{\mu\nu} - X_{\nu\mu})$

$$\boxed{\mathcal{L}_{tot} = -1/4 F_{\mu\nu} F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - m^2 \phi \bar{\phi}}$$

Kinetic term with two derivatives

3.4. The QED Lagrangian

$\psi(x) \to e^{i\alpha(x)} \psi(x)$ $\qquad A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$

symmetry of Dirac Lagrangian for $\alpha(x) = \text{const.}$

$D_\mu = \partial_\mu + ieA_\mu$

$$\boxed{\mathcal{L}_{QED} = -1/4 F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi}$$

3.5. Quantisation of A_μ

$\mathcal{L} = -1/4 F_{\mu\nu} F^{\mu\nu}$ will not change under

$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ (gauge transformation)

This redundancy is a problem for quantisation!

Remove it by gauge fixing

① Lorentz gauge $\quad \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = 0$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu} \qquad (1)$$

still not unique

② Coulumb gauge $A'^0 = A^0 + \underbrace{\partial^0\Lambda}_{\frac{\partial}{\partial t}\Lambda} = 0$

$\Rightarrow \boxed{\frac{\partial}{\partial t}\Lambda = -A^0}$ together with (1) we have

$$A_0 = A^0 = 0 \quad \text{and} \quad \partial_i A^i = 0$$

reduces from 4 degrees of freedom in A_μ to 2

Quantisation:

1.) conjugate momentum for A_μ:

$$\pi^0 = \frac{\delta \mathcal{L}}{\delta \dot{A}_0} = 0 \qquad \pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = -\dot{A}^i + \partial^i A^0 = F^{0i} = E^i$$

Electric field strength

2.) canonical commutator:

$$[A^i(\vec{x}), E^j(\vec{y})] = i\int \frac{d^3k}{(2\pi)^3}\left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) e^{i\vec{k}(\vec{x}-\vec{y})}$$

becomes the $\delta^{ij}\delta(\vec{x}-\vec{y})$ we know   new! why?

because $\partial_i A^i = 0$

and therefore $[\partial_i A^i(\vec{x}), E^j(\vec{y})] = \frac{\partial}{\partial x^i}[A^i(\vec{x}), E^j(\vec{y})]$

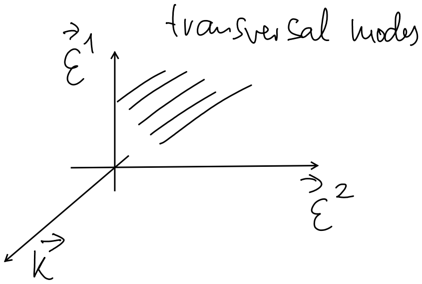
(EX 2.2) requires the additional term = 0

3.) Mode expansion:

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{\lambda}(k)\left[a_k^\lambda e^{-ikx} + (a_k^\lambda)^\dagger e^{ikx}\right]$$

polarisation vectors

$$\vec{\nabla}\vec{A} = \partial_i A^i = 0 \rightsquigarrow \boxed{\vec{k}\cdot\vec{\varepsilon}^{\lambda} = 0}$$



$\left(\vec{\varepsilon}^{\,\lambda}\cdot\vec{\varepsilon}^{\,\varrho}=\delta^{\lambda\varrho}\right)$

with $\left[a^{\lambda}_{\vec{k}},\,(a^{\varrho}_{\vec{k}'})^{+}\right]=2k_0(2\pi)^3\delta^{\lambda\varrho}\,\delta(\vec{k}-\vec{k}')$

again creation and annihilation operators of HO

4.) Hamiltonian

$$H=\frac{1}{2}\int d^3x\left(\dot{\vec{A}}^2+(\vec{\nabla}\times\vec{A})^2\right)$$

$$\vdots$$

$$=\sum_{\lambda}\int\frac{d^3k}{(2\pi)^3k_0}\,\frac{k_0}{2}\left((\vec{a}^{\,\lambda}_{\vec{k}})^{+}\,\vec{a}^{\,\lambda}_{\vec{k}}+\text{vacuum energy}\right)$$