Euler - Lagrange equation:

$$S = \int dt \int_R d^3x \; \mathcal{L}(\phi, \underbrace{\dot{\phi}, \nabla\phi}, t)$$

$$\partial_0 = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x^i} \qquad \partial_\mu \text{ with } \mu = 0, 1, 2, 3$$

$$\delta S = \int dt \int_R d^3x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi \right)$$

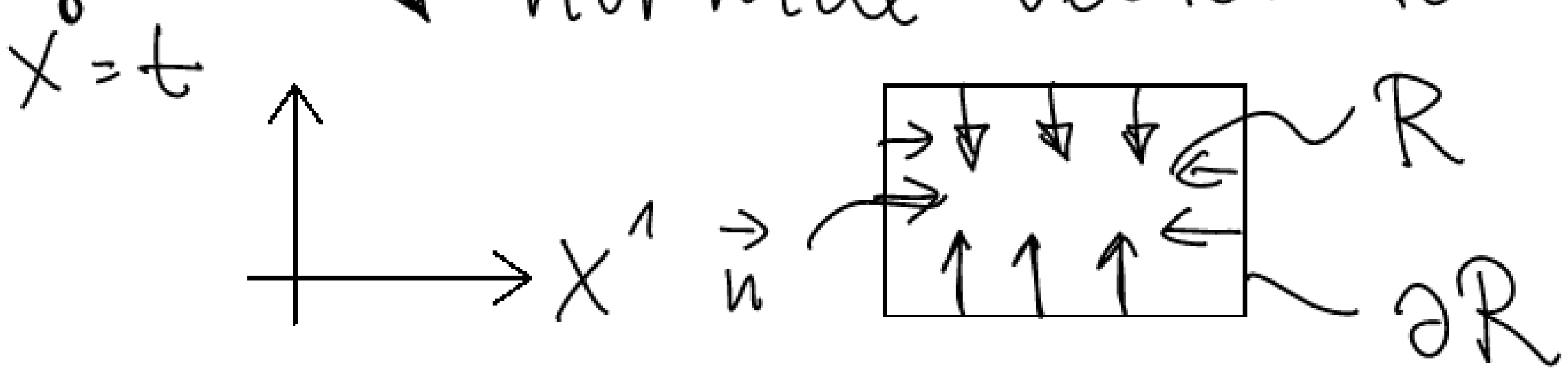
$$\overset{\text{i.b.p}}{=} \int dt \int_R d^3x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi = 0$$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0} \quad + \text{boundary conditions}$$

⚡ Integration by parts now i.e. requires

$$n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{\partial\tilde{R}} = 0 \quad \text{boundary of space time region}$$

n_μ normal vector to the boundary



two possibilities

a) von Neumann boundary condition (BC)

$$n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Big|_{\partial R} = 0 = n_\mu \partial^\mu \phi \Big|_{\partial R}$$

normal derivative set to zero at boundary

for Klein-Gordon field

b) Dirichlet BC $\quad \delta\phi\big|_{\partial R} = 0$

c) mix between a) and b)

Example:

$$\left.\begin{aligned}
\mathcal{L} &= \frac{1}{2}\Big(\partial_\mu \phi \underbrace{\partial^\mu \phi}_{= \eta^{\mu\nu}\partial_\nu \phi} - m^2 \phi^2 \Big) \\
&\text{with } \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \\
&\rightsquigarrow \partial^0 = \frac{\partial}{\partial t}, \quad \partial^i = -\frac{\partial}{\partial x^i}
\end{aligned}\right\} \text{reproduces KG Lagrangian from first lecture}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial_\nu \phi \, \eta^{\nu\mu} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

Field equations:

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = -\Box \phi + m^2 \phi = 0}$$

3. Symmetries & conservation Laws

3.1. Motivation

Question: Can we change field(s) ϕ such that Lagrangian $\mathcal{L}$ does not change?

We already know $\delta \mathcal{L} = 0$

$$\Longleftrightarrow \underbrace{\left[-\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \frac{\partial \mathcal{L}}{\partial \phi} \right]}_{\text{field equations}} \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}_{\text{boundary term}}$$

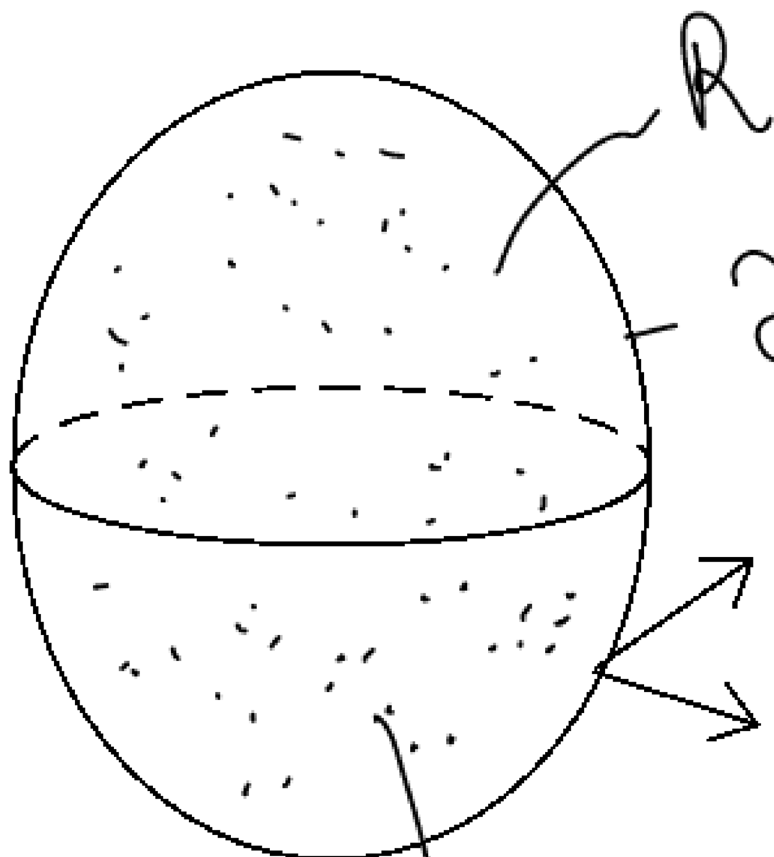
$$\rightsquigarrow \quad J^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \quad \text{with}$$

$$\boxed{\underbrace{\partial_\mu J^\mu = 0}_{\text{conserved current}}} \quad + \text{ field equations } \Longleftrightarrow \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

$J^0 = \rho$, $\quad J^i = j^i$ with $\vec{j} = (j^1, j^2, j^3)$
and $\nabla = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \quad \text{continuity equation}$$

$$\int_R dV \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \right) = \frac{\partial Q}{\partial t} + \int_{\partial R} d\vec{S} \cdot \vec{j} = 0$$



R ~ region
∂R ~ region's boundary
$\vec{j}$ ~ current
$d\vec{S}$ ~ normal vector on boundary

think i.e. about electrically charged particles

charge $Q = \int_R dV \rho$ ~ charge density

change of charge over time = − current through boundary

Refinement: Symmetries which act on space & time like $\delta x^\mu = \xi^\mu$

$$\rightsquigarrow \quad \delta \mathcal{L} = \xi^\mu \partial_\mu \mathcal{L} \neq 0 \qquad \text{(shift)}$$

and therefore: $\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \xi^\mu \mathcal{L}}$

Example: $\delta\phi = \xi^\mu \partial_\mu \phi$

$$J^\mu = \xi^\nu \Big(\underbrace{\partial^\mu \phi \, \partial_\nu \phi - 1/2 \, \delta^\mu_\nu \, \partial^\rho \phi \, \partial_\rho \phi + 1/2 \, \delta^\mu_\nu m^2 \phi^2}_{T_\nu{}^\mu} \Big)$$

$T_0{}^0 = 1/2 \left[\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2 \right] \hat{=}$ density

$T_0{}^i = - \dot\phi \, \vec\nabla^i \phi \hat{=}$ current

conserved charge $H = \int dV \; T_0{}^0 =$ energy

remember Hamiltonian $H = T + U$ vs.

Lagrangian $\mathcal{L} = T - U$

now we also have $T_0{}^0 =$ Hamiltonian ($\hat{=}$ energy) density

"kinetic energy" $T = \int dV \, \frac{1}{2} \dot\phi^2$

"potential energy" $U = \int dV \, \frac{1}{2} \left[(\nabla\phi)^2 + m^2\phi^2 \right]$

for $T_j{}^0$ we find momentum conservation!