Selected Tools of Modern Theoretical Physics 2B

symmetries in physics ⇝ Lie groups

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lectures: Wed. 12:15 – 14:00 511
tutorials: Mon. 10:15 – 12:00 422
} usually

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- lecture notes, exercise sheets, and recordings @

https://www.fhassler.de/teaching/ss_25/sel_tools_theory_2B

- online ~ 1 week before the tutorial
- assigned at Wed. 21:00 – " –

Will be graded.

EXAM
- written exam @ end of semester
- at least 50% of the points of assigned problems to qualify

⇝ check your points at the website

remote participants need to upload solutions before the tutorial

1. Introduction

1.1. Why do we care?

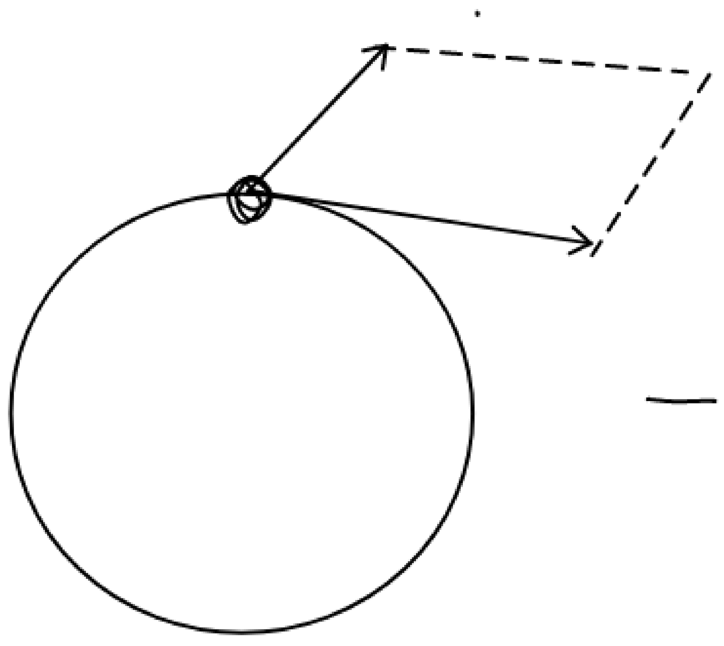
- (Lie) groups characterise symmetries in physical systems

classical mechanics: - help to solve equations of motion

quantum mechanics: - quantisation of angular momentum → Lie group $SU(2)$
- quantum numbers l and m

quantum field theory
- Lorentz and Poincaré group → spin of particles
- (gauge) groups characterise particle content of the standard model

- Lie Groups parameterised by continous set of variables = coordinates on manifold



tangent space = Lie algebra
- encodes already most aspects of the Lie group
- much easier to deal with (just linear algebra)

- complete classification of semi simple Lie algebras (later in the course)

- Lie groups / algebras are not directly visible in physics
→ we see only representations

i.e. Lorentz group, $SO(3,1)$, acts

- on Scalars (trivially) ϕ ← Higgs boson
- on Vectors V^μ ← boson, like photon
- on Spinor Ψ

<u>representations</u> decompose into fundamental building blocks = irreducible repr. = <u>irreps</u>

1.2. Rotations in 3 dim. as example

defined by: $R^T \cdot R = \mathbb{1}_3$, $\det R = 1$ ⇝ $SO(3)$

$R \in M_{3\times 3}(\mathbb{R})$, real 3x3 matrices

like: $R_1(\alpha_1)\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

$R_2(\alpha_2) = \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}$, and $R_3(\alpha_3) = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

a general rotation can be written as:

$R = R_1(\alpha_1)\, R_2(\alpha_2)\, R_3(\alpha_3)$

check: 3x3 matrix R has $3\cdot 3 = 9$ real parameters

$(R^T \cdot R)^T = (\mathbb{1}_3)^T = R^T R = \mathbb{1}_3$

↳ $\frac{1}{2}\cdot 3(3+1) = 6$ constraints

$\det R = 1$ $\det(R^T \cdot R) = \det(\mathbb{1}_3) = 1 = \det R^T \det R = (\det R)^2$

⇝ $\det R = \pm 1$ discrete choice

$\underbrace{9}_{\text{param.}} - \overbrace{6}^{\text{constr.}} = 3$ ⟵ dimension of <u>Lie group $SO(3)$</u>

$\underbrace{SO(3)}_{\det R = 1} \subset \underbrace{O(3)}_{\det R = \pm 1}$

special orthogonal group

1.2.1. so(3) Lie algebra

tangent space of SO(3) at the identity element

$\mathbb{1}_3 = R(0,0,0) = R(\vec{0})$

spaned by $\boxed{E_i = \frac{\partial}{\partial \alpha_i} R(\vec{\alpha})\Big|_{\vec{\alpha}=0}}$

$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, and $E_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

They span the vector space of antisymmetric, real 3x3 matrices

check: $E^T = -E$ $\quad \frac{1}{2} 3(3+1) = 6$ constraints

$9 - 6 = 3$ dimensional ✓

Lie algebra = Vector space V + product $V \times V \to V$

naive product would be matrix product

$E_1 \cdot E_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ↯ not anti symmetric

⇝ correct product is the commutator

$[E_i, E_j] = E_i \cdot E_j - E_j \cdot E_i = \sum_{k=1}^{3} \varepsilon_{ijk} E_k$

Levi-Civita symbol, totally anti-symmetric with $\varepsilon_{123} = 1$

$[E_1, E_2] = \overset{=1}{\varepsilon_{123}} E_3$

$[E_3, E_1] = \varepsilon_{312} E_2 = \overset{=1}{\varepsilon_{123}} E_2$

$[E_2, E_3] = \varepsilon_{231} E_1 = \overset{=1}{\varepsilon_{123}} E_1$ //

1.2.2. Outlook representations

Can we find other (real) matrices E_i' with the same commutators?

Yes! I.e. product representation

$$E_i' = E_i \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes E_i$$

Kronecker product with

≙ coupled angular momentum in QM

$so(3) = su(2)$

$$A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & & \vdots \\ a_{n1} B & \cdots & a_{nn} B \end{pmatrix} \qquad A \in M_{n \times n}, \quad B \in M_{m \times m}$$

$$A \otimes B \in M_{n \cdot m \times n \cdot m} \rightarrow E_i' \in M_{9 \times 9 = 3 \cdot 3}$$

we can check:

$$[E_i', E_j'] = [E_i, E_j] \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes [E_i, E_j]$$
$$= \sum_{k=1}^{3} \varepsilon_{ijk} E_k'$$

<u>BUT</u> not an irrep! $\rightsquigarrow$ use Casimir operator

$$C := -E_1'^2 - E_2'^2 - E_3'^2 ≙ \vec{L}^2 \text{ in QM}$$

1. find basis in which C is diagonal
2. transform E_i' into this basis

$$C' = S^{-1} C S = \mathrm{diag}(\underbrace{6, \dots, 6}_{5\times}, \underbrace{2, \dots, 2}_{3\times}, 0)$$

$$E_i'' = S^{-1} E_i' S = \begin{pmatrix} \boxed{5\times5} & 0 & 0 \\ 0 & \boxed{3\times3} & 0 \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

irreps

<u>We say</u>: $3 \times 3 \rightarrow 1 + 3 + 5$