Quantum Field Theory

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office: 448

lectures: Fri. 10:15 - 12:00
tutorials: Fri. 08:15 - 10:00
447

exercises & handwritten notes at

https://www.fhassler.de/teaching

- online ~1 week before tutorial
- assigned at Sun. 21:00 –"– ⇝ email

Will be graded.

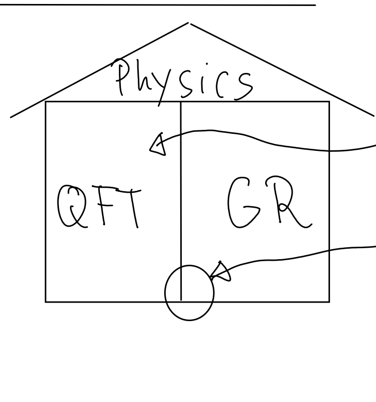
Problems? Contact me or course assistant
Alex Swash (alex.swash@outlook.com)

$\boxed{\text{exam}}$

- in written @ end of semester
- at least 50% points of assigned exercise problems to qualify

office hours: Tue. 15:00 - 17:00

0. Motivation



very well tested

unification i.e. string theory

1. Canonical quantisation

1.1. Klein-Gordon Field: Lagrangian

It is governed by the action

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$ Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\nabla}\phi)^2 - \tfrac{1}{2}m^2\phi^2 \qquad \dot{\phi} = \frac{\partial}{\partial t}\phi = \partial_t \phi$$

$$\vec{\nabla} = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$$

better in covariant form with metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}$$

$$\boxed{\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi\right) - \tfrac{1}{2}m^2\phi^2}$$

equation of motion from principle of least action

$$\delta S = 0$$

$$\delta S = \int d^4x \left[\cancel{\tfrac{1}{2}}\cancel{2}\left(\partial_\mu \delta\phi \, \partial^\mu \phi\right) - \cancel{\tfrac{1}{2}}\cancel{2}m^2 \phi \, \delta\phi \right]$$

derivatives can be "removed" by integration by parts

$$\int d^4x \, \partial_\mu f_1 \, f_2 + \int d^4x \, f_1 \, \partial_\mu f_2 = \int d^4x \, \partial_\mu (f_1 f_2) = 0$$

we ignore boundary terms (at the moment)

$$\delta S = \int d^4x \left[-\partial_\mu \partial^\mu \phi - m^2 \phi \right] \delta\phi = 0$$

$$\Rightarrow \boxed{\underbrace{\partial_\mu \partial^\mu}_{\Box} \phi + m^2 \phi = 0}$$ Klein-Gordon equation

1.2. Hamiltonian Formulation

remember: classical mechanics positions q^i

$\updownarrow$

conjugate momenta $p_i = \dfrac{\partial L}{\partial \dot{q}^i}$

Hamiltonian $H(t) = \sum_i p_i \dot{q}^i - L$ ← Lagrangian

field theory $S = \int dt\, L = \int dt \int d^3x\, \mathcal{L}(\phi, \dot{\phi})$

field

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)} \quad \text{conjugate momentum to } \phi$$

$$H(t) = \int d^3x \left[\pi \dot{\phi} - \mathcal{L}\right] = \int d^3x\, \mathcal{H}$$

Hamiltonian density

For the Klein-Gordon Lagrangian:

$$\pi(x) = \frac{\delta}{\delta \dot{\phi}} \left(\tfrac{1}{2} \dot{\phi}^2 - \tfrac{1}{2} (\vec{\nabla}\phi)^2 - \tfrac{1}{2} m^2 \phi^2 \right) = \dot{\phi}$$

$$H = \int d^3x \left[\tfrac{1}{2} \pi^2 + \tfrac{1}{2} (\nabla \phi)^2 + \tfrac{1}{2} m^2 \phi^2 \right]$$

Field equations:

1.) Hamiltonian ✓
2.) Poisson brackets (Pb's)
3.) Time evolution

2.) $\{ \phi(t, \vec{x}), \pi(t, \vec{y}) \} = \delta(\vec{x} - \vec{y})$

equal time

all other Pb's are 0

3.) for all functions of ϕ and π, $O(\phi, \pi)$, we have

$$\boxed{\frac{\partial}{\partial t}\mathcal{O} = \{\mathcal{O}, H\}}$$

Let's check for $\mathcal{O}(\phi, \pi) = \phi$:

$$\frac{\partial}{\partial t}\phi(t,\vec{x}) = \int d^3y \,\{\phi(t,\vec{x}), \tfrac{1}{2}\pi^2(t,\vec{y})\}$$

$$= \int d^3y \underbrace{\{\phi(t,\vec{x}), \pi(t,\vec{y})\}}_{\delta(\vec{x}-\vec{y})} \pi(t,\vec{y})$$

$$= \pi(t,\vec{x}) //$$

$$\frac{\partial}{\partial t}\pi(t,\vec{x}) = \ldots = (\vec{\nabla}^2 - m^2)\,\phi(t,\vec{x})$$

$$\ddot{\phi} = \dot{\pi} = (\vec{\nabla}^2 - m^2)\,\phi$$

$$\ddot{\phi} - \vec{\nabla}^2\phi + m^2\phi = \underline{\underline{\partial_\mu \partial^\mu \phi + m^2\phi = 0}}$$

Klein-Gordon equation

1.3. Quantisation

Fourier transformation $\rightsquigarrow$ momentum space

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{p}}\,\phi(t,\vec{p})$$

Klein-Gordon eq. $\left[\frac{\partial^2}{\partial t^2} + \underbrace{(|\vec{p}|^2 + m^2)}_{\omega_{\vec{p}}^2}\right]\phi(t,\vec{p}) = 0$

$$\omega_{\vec{p}} = \sqrt{|\vec{p}| + m^2}$$

$\Rightarrow$ Harmonic oscillators with frequency $\omega_{\vec{p}}$

first one HO

$$\left.\begin{aligned} H_{HO} &= \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2\phi^2, \quad \phi = \frac{1}{\sqrt{2\omega}}(a + a^+) \\ p &= -i\sqrt{\frac{\omega}{2}}(a - a^+) \end{aligned}\right\} \text{with } [\phi, p] = i\hbar, \ \hbar = 1$$

implies $\quad [a, a^+] = 1$ raising operator

lowering operator

g

Hilbert space: $\quad a|0\rangle = 0 \quad$ vacuum or ground state

$$(a^+)^n |0\rangle = |n\rangle$$

$$\Rightarrow \quad H_{Ho} = \omega\left(a^+ a + 1/2\right) \qquad a^+ a|n\rangle = n|n\rangle$$

$$H_{Ho}|n\rangle = \omega(n + 1/2)|n\rangle$$

next KG theory

usually supressed

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}(t) + a^+_{-\vec{p}}(t)\right) e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^+_{-\vec{p}}\right) e^{i\vec{p}\cdot\vec{x}}$$

with $[a_{\vec{p}}, a^+_{\vec{p}'}] = (2\pi)^3 \delta(\vec{p} - \vec{p}')$ from

$[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta(\vec{x} - \vec{y}) \quad$ (please check)

Comparing with $\{\phi(\vec{x}), \pi(\vec{y})\} = \delta(\vec{x} - \vec{y})$ we find

$$\boxed{\{\cdot,\cdot\} \to i\underbrace{\hbar}_{1 \text{ for us}}[\cdot,\cdot]} \qquad \text{canonical quantisation}$$

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a^+_{\vec{p}} a_{\vec{p}} + \underbrace{1/2\,[a_{\vec{p}}, a^+_{\vec{p}}]}_{\infty = \text{vacuum energy}}\right)$$

Spectrum:

vacuum state

$a_{\vec{p}}|0\rangle = 0$

$a^+_{\vec{p}}|0\rangle =$ 1-particle state with momentum $\vec{p}$

and energy $E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$
(remember $c = 1$)

1.4. Heisenberg picture & propagator

$$\phi(x) = e^{iHt}\, \phi(\vec{x})\, e^{-iHt}$$

4-position $x^\mu = (\underset{= t}{x^0}, \underbrace{x^1, x^2, x^3}_{\vec{x}})$

same for $\pi(x)$

now we have : $\quad i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]$

(compare with) $\quad \frac{\partial}{\partial t} \mathcal{O} = \{\mathcal{O}, H\}$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip_\mu x^\mu} + a^\dagger_{\vec{p}}\, e^{ip_\mu x^\mu} \right) \Big|_{p^0 = E_{\vec{p}}}$$

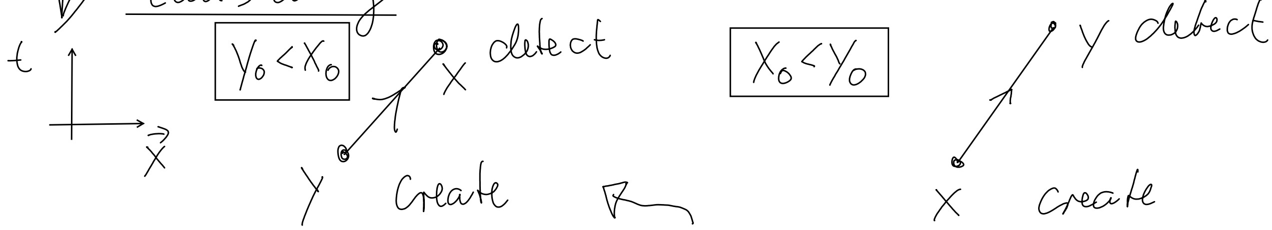
on-shell

$$\pi(x) = \frac{\partial}{\partial t} \phi(x) = \dot{\phi}$$

Propagator: create particle at position y and
EXPERIMENT detect it at position x

$$D(x-y) = \langle 0 |\, \phi(x)\, \phi(y)\, | 0 \rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{-ip(x-y)}$$

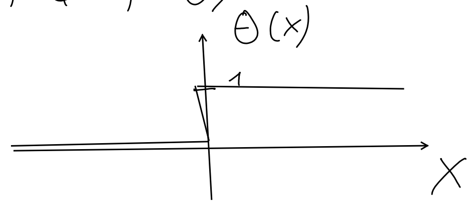
↯ Causality



$$D_F(x-y) = \Theta(x^0 - y^0) \langle 0 |\, \phi(x)\, \phi(y)\, | 0 \rangle +$$
Feynman

$$\Theta(y^0 - x^0)\, \langle 0 | \phi(y)\, \phi(x) | 0 \rangle$$

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$



$$= \lim_{\varepsilon \to 0^+} \mp \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau \pm i\varepsilon}\, e^{\mp i x \tau}\, d\tau$$ ↗ EX 2.1.

$$\boxed{D_F(x-y) = \lim_{\varepsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x-y)}}$$

Feynman propagator for KG field

$$D_F(x-y) = \Theta(x^0 - y^0)\, D(x-y) + \Theta(y^0 - x^0)\, D(y-x)$$
$$= \langle 0 | T\, \phi(x)\, \phi(y) | 0 \rangle$$

↖ Time ordering operators
"later" operators go to the left

2. Dirac Field

2.1. Lorentz transformations

elements of Lorentz group $O(3,1)$

coordinates: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

such that $x^\mu x^\nu g_{\mu\nu} = x'^\mu x'^\nu g_{\mu\nu}$

scalar fields: $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$

vector fields: $V^\mu(x) \to V'^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x)$

example $\partial^\mu \phi(x) = V^\mu(x)$

dual, 1-form fields : $A_\mu(x) \to A'_\mu(x) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}x)$

then $A_\mu V^\mu$ is a scalar

example $A_\mu(x) = \partial_\mu \phi(x)$

Questions: • Are there more examples? Yes there are!

• How do we classify them?

↗ Monographic lecture "Lie algebras & Lie groups"

so(3,1) Lie algebra

$\frac{1}{2}\cdot 4(4-1) = 6$ generators $J^{\mu\nu} = -J^{\nu\mu}$

defined by commutators:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)$$

1) $J_1 = J^{23}$, $J_2 = J^{31}$, $J_3 = J^{12}$ generate rotations of 3 spacial dir.

2) $K_1 = J^{01}$, $K_2 = J^{02}$, $K_3 = J^{03}$ boosts

$$[J_i, J_j] = i\sum_{k=1}^{3} \varepsilon_{ijk} J_k = i\,\varepsilon_{ijk} J_k \qquad so(3) \text{ subalgebra}$$

$[J_i, K_j] = i\,\varepsilon_{ijk} K_k$ and $[K_i, K_j] = -i\,\varepsilon_{ijk} J_k$

Task: Find explicit representations for matrices $J^{\mu\nu}$!

2.2. γ-matrices and the Dirac algebra

More concret: find 4 $n\times n$ matrices γ^μ with

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} \cdot \mathbb{1}_{n\times n}$$ ← $n\times n$ identity matrix

$\boxed{\text{EX } 1.2}$ Check that they generate $so(3,1)$!

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \sigma^i = \text{Pauli matrices}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We also need one more γ-matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix} \text{ with } \{\gamma^5, \gamma^\mu\} = 0$$

and therefore $[\gamma^5, J^{\mu\nu}] = 0$

γ-matrices act on 4-component vectors

$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ called <u>Dirac-Spinors</u>

They decompose into 2 fundamental irreps of $so(3,1)$, the 2-component (but complex) Weyl spinors ψ_L & ψ_R. They are the ∓ 1 eigenvalues of γ^5.

Contraction of two Dirac-spinors ⇝ Lorentz scalar

naively: $\Psi^\dagger \Psi$ (like for vector) does not work!

rather $\bar{\Psi}\Psi$ with

$$\boxed{\bar{\psi} = \psi^{\dagger}\gamma^0}$$ Dirac conjugation

EX 1.2 verify that $\bar{\psi}\psi$ is a Lorentz scalar

2.3. Dirac equation

$$\boxed{S_{Dirac} = \int d^4x \; \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi}$$

notation $\gamma^{\mu}\partial_{\mu} = \cancel{\partial}$
or $\gamma^{\mu}p_{\mu} = \cancel{p}$

field equation: $\frac{\delta S_{Dirac}}{\delta \bar{\psi}} = 0$

results in $\boxed{(i\cancel{\partial} - m)\psi = 0}$ = Dirac equation

plane wave solutions

$$\psi(x) = u(p)\, e^{-ipx} + v(p)\, e^{ipx}, \quad p^0 > 0$$

$\hookrightarrow$ $(\cancel{p} - m)\, u(p) = 0$ and $(\cancel{p} + m)\, v(p) = 0$

both have two linearly independent solutions

$u(p) = u^s(p)$ $s = 1, 2$ and $v(p) = v^r(p)$ $r = 1, 2$

which can be normalised to

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta^{rs} \qquad \bar{u}^r(p)\, v^s(p) = 0$$

$$\bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs} \qquad \bar{v}^r(p)\, u^s(p) = 0$$

Interpresation of them as electrons and positrons (see EX 1.3)

2.4. Quantisation of the Dirac field

conjugate momentum to ψ is $i\psi^{\dagger}$

Hamiltonian: $H = \int d^3x \; \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$

spacial part only

mode expansion:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}} u^s(p) e^{-ipx} + (b^s_{\vec{p}})^\dagger v^s(p) e^{ixp} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(b^s_{\vec{p}} \bar{v}^s(p) e^{-ipx} + (a^s_{\vec{p}})^\dagger \bar{u}^s(p) e^{ipx} \right)$$

$$\{a^r_{\vec{p}}, (a^s_{\vec{q}})^\dagger\} = \{b^r_{\vec{p}}, (b^s_{\vec{q}})^\dagger\} = (2\pi)^3 \delta(\vec{p}-\vec{q}) \delta^{rs}$$

⚡ not Poisson brakets, but anti-commutator !

$$\{a,b\} = a\cdot b + b\cdot a$$

all other $\{\cdot,\cdot\} = 0$

reason for $\{\circ,\circ\}$ instead of $[\circ,\circ]$ is that we are dealing with fermions

vacuum $|0\rangle$ annihilated by

$$a^s_{\vec{p}} |0\rangle = b^s_{\vec{p}} |0\rangle = 0$$

We can only have on particle with given state

$(a^1_{\vec{p}})^\dagger |0\rangle$ because $(a^1_{\vec{p}})^\dagger (a^1_{\vec{p}})^\dagger |0\rangle = 0$

$$\{a^1_{\vec{p}}, a^1_{\vec{p}}\} = 2(a^1_{\vec{p}})^2 = 0$$

= Pauli exclusion principle !

Hamiltonian: $H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$

and Feynman propagator:

$$D_F(x-y) = \langle 0 | T\, \psi(x) \bar{\psi}(y) | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$