An Introduction to String Theory

by Falk Hassler
email: falk.hassler@uwr.edu.pl
office: 448 ; office hours: Thu 14^{00} - 16^{00}
lectures: Thu 8^{15} - 10^{15} (2 ½ accademic hours)
tutorials: Thu 10^{30} - 12^{30} (— " —)

- both in 447
- assistant MSc Luca Scala
  email: 339123@uwr.edu.pl

exercises & handwritten notes on the website

https://www.fhassler.de/teaching#st_2022

problems appear ≈ 1 week before the tuorial

TODO: i) register @ website with USOS
optional ii) indicate preferences each week
optional iii) check how much points you got
manditory iv) prepare your assigned problems

You can be absent (without attest) 2x
but have to indicate it on the website before assignments are made.

exam:
- in written at the end of the semester
- tutorials are graded
- need ≥ 3 ($> 50\%$) in tutorial to qualify for the exam

1. What, why and how (is/works String Theory)

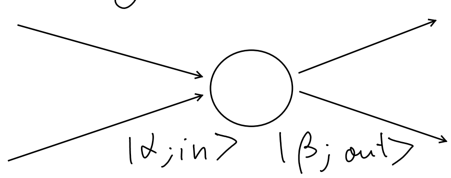
$\underbrace{\text{String Theory}}_{ST}$

1.1. What

$$\boxed{\text{ST is a } \underline{\text{relativistic}}, \ \underline{\text{quantum-mechanical}} \text{ (physical) theory of massless } \underline{\text{fundamental strings}}.}$$

↳ quantum-mechanical: observers make probabilistic predictions about outcome of experiments using Hilbert spaces $\mathcal{H} \ni |\Psi\rangle$ for particle states and Hamiltonians H for time evolution.

example: scattering experiment

prepare $|\alpha\rangle$  register $|\beta\rangle$

theory: $$|\alpha\rangle = \lim_{t\to -\infty} e^{-iHt/\hbar} |\alpha; in\rangle$$

$$|\beta\rangle = \lim_{t\to \infty} e^{-iHt/\hbar} |\beta; out\rangle$$

S-matrix $$(S_{\beta\alpha}) = |\langle \beta; out | \alpha; in\rangle|^2$$

$$= \text{probability } (\alpha \to \beta) \geq 0$$

unitarity

relativistic: different (inertial) observers compare i.e. $S_{\beta\alpha}$ with the help of coordinates X^μ and their transformations

$$X^{\mu} \longrightarrow X^{\mu\prime} \qquad X^{\mu\prime} = \Lambda^\mu{}_\nu X^\nu + a^\mu$$

$\mu = 0, 1, \dots, d-1$

Lorentz transformations

translations

$$\underset{\in O(d-1,1)}{\Lambda^{\rho}{}_{\mu}}\, \eta_{\rho\lambda}\, \Lambda^{\lambda}{}_{\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \dots, +1)$$

Lorentz rotations + translations = Poincaré group

fundamental strings 1-dim. compact object

only characterised by its tension T, $[T] = E^2$

dimension of T = Energy^2

or area (scale) $\sqrt{\alpha'}$ = length / time scale $= l_s$

$$\boxed{T = \frac{1}{2\pi\alpha'} \qquad \alpha' = l_s^2}$$

⇝ There are open and closed strings, which can be orientated & unorientated. They sweep out the **world sheet** as they propagate through space & time.

Action ~ relativistic area $A = \int_\Sigma d^2\sigma \det \partial_\alpha X^\mu \partial^\alpha X_\mu$

$S = T \cdot A$

$\boxed{\text{We work out details \& consequences in this course.}}$

1.2. Why

① Quantum gravity (QG)

Classical gravity, described by general relativity (GR) is notoriously hard to quantise with traditional methods.

i) perturbatively

$$S_{E-H} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R + \text{matter}$$

$g = \det(g_{\mu\nu})$

dim. analysis $\quad [g] = L^2 = M^{-2}$

$$[R] = L^{-2}$$

$$\rightsquigarrow [G_N] = M^{-2} = L^2$$

$G_N = \frac{1}{M^2_{Planck}}$ with $M_{Pl} = 1{,}2 \cdot 10^{19}$ GeV

$L_{Pl} = 1{,}6 \cdot 10^{-35}$ m

now expand $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu}$ ← small fluctuations

$$S_{E-H} = \int d^4x\, (\partial h)^2 + \frac{1}{M_{Pl}} h(\partial h)^2 + \frac{1}{M_{Pl}^2} h^2 (\partial h)^2 + \dots$$
$$+ \int d^4x \frac{1}{M_{Pl}} h_{\mu\nu} T^{\mu\nu}$$

suppressed indices

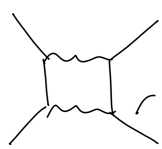
energy-momentum tensor $\quad T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi$

scalar field → matter

QFT course: cancel divergences with counter terms

graviton

$$\sim \int^\Lambda \frac{d^4k}{k^4} \cdot \frac{k^2}{M_{Pl}} \sim \frac{\Lambda}{M_{Pl}^2}$$



scalar

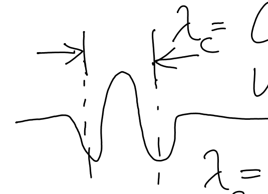
$$\sim \int \frac{d^4k}{k^8} \frac{k^8}{M_{Pl}^4} \sim \frac{\Lambda^4}{M_{Pl}^4}$$

⋮

↳ ∞ many counterterms required with unknown couplings $\rightsquigarrow$ non-renormalisable

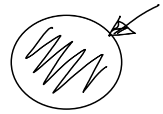
ii) non-perturbatively

- particle in QM wave packet



λ_c = Compton-wave length

$$\lambda_c = \frac{h}{M \cdot c}$$

- black hole :  r_S = Schwarzschild radius

$$r_S = \frac{2 G_N M}{c^2}$$

for M_{Pl}: $\lambda_c = \pi r_S$

$$\rightarrow M_{Pl} = \sqrt{\frac{\hbar c}{G_N}}$$

↪ Theory breaks down @ M_{Pl}

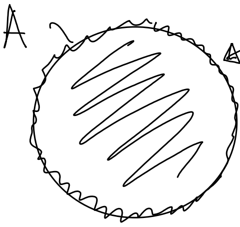
2. String theory facts

- $\mathcal{H}_{string} > \mathcal{H}_{gravity}$
- tree-level (no-loops), low-engery ST interactions equal to those of gravitons (much more later)
- loop amplitudes are finite (no divergences)

  $l_S \sim l_{Pl}$ ST becomes cadidate for QG

- (completing of) string theory realises the <u>holographic principle</u>

$$S_{BH} = \frac{A}{4 L_P^2}$$

A  degrees of freedom on the surface not in the bulk

→ fundamental feature of QG

↗ AdS / CFT correspondence

- $\mathcal{H}_{string} > \mathcal{H}_{gauge\ theory}$ ← i.e. might describe $SU(3) \times SU(2) \times U(1)$ Standard model

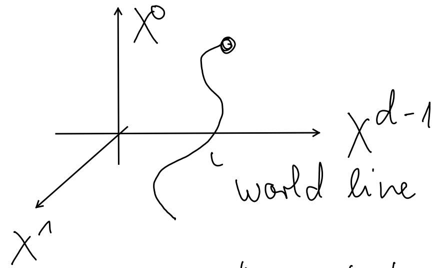
↳ has room for Higgs, GUT and dark matter

Problem: No experimental evidences for strings or Supersymmetry (ST's side kick)

2. Relativistic actions

2.1. Relativistic point particle

Consider free particle of mass m moving through Minkowski space $X^0, \dots, X^{d-1}$



Question: Which trajectories are physical?

Answer: Straight, time-like lines

= "Shortest possible pathes" where length is measured in proper time

$X^\mu(\tau)$, $\tau \in [\tau_i, \tau_f]$

$$S(X(\tau)) = \underset{\uparrow \text{ length}}{l} = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\dot{X}^2}, \quad \dot{X}^2 = \left(\frac{\partial}{\partial\tau} X^\mu\right)\left(\frac{\partial}{\partial\tau} X_\mu\right)$$

Equation of motion: $\frac{\delta S}{\delta X^\mu} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\right) = 0$, $\mathcal{L} = -m\sqrt{-\dot{X}^2}$ ← Lagrangian

Euler - Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = P_\mu = m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$$

$$\frac{d}{d\tau} P_\mu = 0$$

further features

1) reparamitrisation invariant
≙ "gauge degree of freedom" like in E-M

$\tau = \tau(s)$ $\quad (X')^2 = \left(\frac{d\tau}{ds}\right)^2 (\dot{X})^2$

$$S = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\left(\frac{ds}{d\tau}\right)^2 (X')^2} = -m \int_{s_i}^{s_f} ds \sqrt{-(X')^2}$$

$s_{f/i} = s(\tau_{f/i})$ same action

2) Action can be generalised to

a) curved space $\qquad S = -m \int d\tau \sqrt{-\dot{X}^\mu g_{\mu\nu} \dot{X}^\nu}$

$\rightsquigarrow$ equation of motion becomes geodesic equation

b) coupled to E-M field $\qquad S \to S + \int d\tau A_\mu \dot{X}^\mu$

3) does not make sense for massless particles

4) not good for quantisation, i.e. because of $\sqrt{\ldots}$

$\hookrightarrow$ $\boxed{\text{to fix this problem introduce "auxilary" Riemannian metric } \quad h = e^2 d\tau^2, \quad h_{\tau\tau} = e^2 > 0 \text{ on world line}}$

new action:

$$S(X,e) = \frac{1}{2} \int d\tau \underbrace{\sqrt{\det h}}_{e} \cdot \Big(\underbrace{h^{\tau\tau}}_{e^{-2}} \partial_\tau X^\mu \partial_\tau X_\mu - m \Big)$$

$$= \frac{1}{2} \int d\tau \left(\frac{\dot{X}^2}{e} - e m^2 \right)$$

Equations of motion: $\qquad \frac{\delta S}{\delta e} = 0 \to -\frac{\dot{X}^2}{e^2} = m^2$

or $e = \frac{\sqrt{-\dot{X}^2}}{m}$, $\qquad \frac{\delta S}{\delta \dot{X}^\mu} = 0 \to \frac{d}{d\tau}\left(\frac{\dot{X}}{e}\right) = 0$

$$m \frac{d}{d\tau} \frac{\dot{X}}{\sqrt{-\dot{X}^2}} = 0$$ (same as before :-))

further features

1) also reparametrisation invariant with

$\tilde{\tau} = f(\tau), \quad d\tilde{\tau} = f'(\tau)\, d\tau$

$\to \tilde{e} = \frac{e}{f'(\tau)}$ $\qquad$ because $\quad \tilde{h} = \tilde{e}^2 d\tilde{\tau}^2 = \frac{e^2}{\cancel{f'(\tau)^2}} \cdot \cancel{f'(\tau)^2} d\tau^2$

$= h$

and of course $\partial_\tau = f'(\tau)\, \partial_{\tilde{\tau}}$

2) Works for massless ($m=0$) and even tachyonic ($m^2<0$) particles.

3) We may use reparametrisation invariance to fix $\tilde{e}=1$, where

$$S = \frac{1}{2}\int d\tilde{\tau}\ \left(X'^2 - m^2\right)$$

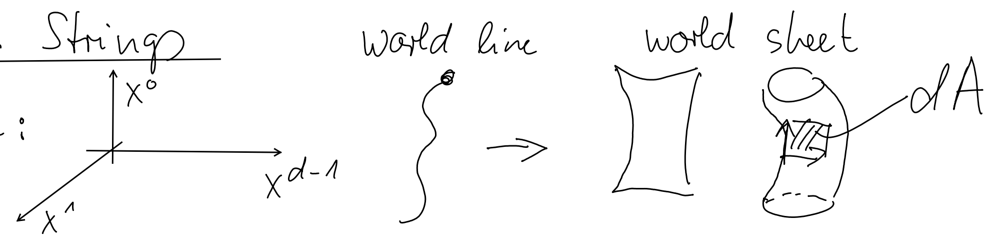
with e.o.m

$$X'' = 0.$$

⚡ But this is not equivalent to the original unless we impose the constraint $X'^2 = -m^2$

2.2. Strings

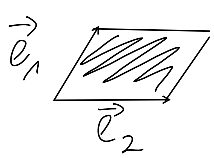
idea:



dictionary: mass m $\longrightarrow$ tension T

proper time $\longrightarrow$ proper area (w.r.t. Minkowski metric)

Riemannian situation:



remember: $|\vec{e}_1||\vec{e}_2|\cos(\sphericalangle(\vec{e}_1,\vec{e}_2)) = \vec{e}_1 \cdot \vec{e}_2$

$$dA = |\vec{e}_1|\cdot|\vec{e}_2|\cdot|\sin(\sphericalangle(\vec{e}_1,\vec{e}_2))|$$

$$= \sqrt{|\vec{e}_1|^2|\vec{e}_2|^2 - (\vec{e}_1\cdot\vec{e}_2)^2}$$ scalar product

$$= \sqrt{\det\begin{pmatrix} \vec{e}_1^{\,2} & \vec{e}_1\cdot\vec{e}_2 \\ \vec{e}_2\cdot\vec{e}_1 & \vec{e}_2^{\,2} \end{pmatrix}}$$

now use $\vec{e}_1 = \frac{d}{d\tau}X^\mu d\tau$, $\quad \vec{e}_2 = \frac{d}{d\sigma}X^\mu d\sigma$

$$A = \int d\tau\, d\sigma \sqrt{\left|\det\begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & X'^2 \end{pmatrix}\right|}$$

$$A = \int d^2\sigma \sqrt{\left| \det \left(\partial_\alpha X^\mu \partial_\beta X_\mu \right)_{\alpha,\beta=0,1} \right|}$$

where $(\tau, \sigma) = (\sigma^0, \sigma^1)$

- $\boxed{\text{The action } S_{NG} = -T \cdot A \text{ is known as the \underline{Nambu-Goto action}}}$

- "physical" String motion time- & space-like tangent vector $\rightsquigarrow$ $\det \leq 0$

  time-: ∂_τ, space-like: ∂_σ

$$S_{NG} = -T \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}$$

Symmetries:

- manifestly Lorentz invariant
- reparameterisation invariant (please check for yourself)