remember: Dirac equation $\boxed{(\not\partial + m)\Psi = 0}$

with $\not\partial := \gamma^\mu \partial_\mu$ γ-matrices spinor $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

original motivation: new representation of

Poincaré-group

Lorentz transformations   translations

in sec. 4.4. generators $J_{\mu\nu} = -J_{\nu\mu}$ $\frac{4\cdot 3}{2} = 6$ ✓

$J_{ij} \hat{=} 3$ rotation
$J_{0i} \hat{=} 3$ boosts
generate Lie algebra

$$\boxed{[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}\right)}$$

on a 4-vector an element $\Lambda^{\mu\nu} J_{\mu\nu}$ acts infinitesimally

by $\delta_\Lambda V^\mu = \Lambda^{\rho\sigma} (J_{\rho\sigma})^\mu{}_\nu V^\nu$ with

$$(J_{\rho\sigma})^\mu{}_\nu = i\left(\eta_{\rho\nu}\delta^\mu_\sigma - \eta_{\sigma\nu}\delta^\mu_\rho\right)$$

$\rightsquigarrow \delta_\Lambda V^\mu = i\left(\Lambda_\nu{}^\mu V^\nu - \Lambda^\mu{}_\nu V^\nu\right)$

Question: Can we do something similar for spinors?

@ $\boxed{S_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]}$ for which we want to

compute $[S_{\mu\nu}, S_{\rho\sigma}] = (I)$

γ-matrix gymnastics:

$$(I) = \frac{1}{2}\cdot\frac{1}{2}\left[\gamma_{[\mu}\gamma_{\nu]}, \gamma_{[\rho}\gamma_{\sigma]}\right] = \frac{1}{4}\left(\gamma_{[\mu}\gamma_{\nu]}\gamma_{[\rho}\gamma_{\sigma]} - \gamma_{[\rho}\gamma_{\sigma]}\gamma_{[\mu}\gamma_{\nu]}\right)$$

so we need:

$$\gamma_\mu \gamma_\nu \gamma_\rho = -\gamma_\mu \gamma_\rho \gamma_\nu + \gamma_\mu \{\gamma_\rho, \gamma_\nu\}$$

Clifford algebra: $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \mathbb{1}$

$$\gamma_\mu \gamma_\nu \gamma_\rho = -\gamma_\mu \gamma_\rho \gamma_\nu + 2\gamma_\mu \eta_{\rho\nu} \quad \text{again!}$$

$$= \gamma_\rho \gamma_\mu \gamma_\nu - 2\eta_{\mu\rho}\gamma_\nu + 2\gamma_\mu \eta_{\rho\nu}$$

$$= \gamma_\rho \gamma_\mu \gamma_\nu - 4\eta_{\rho[\mu}\gamma_{\nu]} \quad //$$

$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = \gamma_\rho \gamma_\mu \gamma_\nu \gamma_\sigma - 4\eta_{\rho[\mu}\gamma_{\nu]}\gamma_\sigma \quad \text{again}$$

$$= \gamma_\rho \gamma_\sigma \gamma_\mu \gamma_\nu - 4\eta_{\sigma[\mu}\gamma_\rho \gamma_{\nu]} - 4\eta_{\rho[\mu}\gamma_{\nu]}\gamma_\sigma$$

$$\Rightarrow \mathrm{I} = -\eta_{[\sigma|\mu}\gamma_{|\rho]}\gamma_\nu + \eta_{[\sigma|\nu}\gamma_{|\rho]}\gamma_\mu - \eta_{[\rho|\mu}\gamma_\nu \gamma_{|\sigma]} + \eta_{[\rho|\nu}\gamma_\mu \gamma_{|\sigma]}$$

$$= 2\eta_{\mu[\rho}S_{\sigma]\nu} - 2\eta_{\nu[\rho}S_{\sigma]\mu}$$

$$= \eta_{\mu\rho}S_{\sigma\nu} - \eta_{\mu\sigma}S_{\rho\nu} - \eta_{\nu\rho}S_{\sigma\mu} + \eta_{\nu\sigma}S_{\rho\mu}$$

⇝ Same relation as for $J_{\mu\nu}$ before

⇝ $S_{\mu\nu}$ generates Lorentz action on spinors

$$\boxed{\delta_\Lambda \psi = \Lambda^{\mu\nu} S_{\mu\nu} \psi}$$

10.3. Lagrangian

We need to combine 2 spinors to a scalar.

Naive: $\delta_\Lambda \psi^\dagger \psi = (\Lambda^{\mu\nu} S_{\mu\nu}\psi)^\dagger \psi + \psi^\dagger \Lambda^{\mu\nu} S_{\mu\nu} \psi$

$$= \Lambda^{\mu\nu} \psi^\dagger (S^\dagger_{\mu\nu} + S_{\mu\nu}) \psi$$

Check: $(S_{0i})^\dagger = \frac{1}{4}[\gamma_0,\gamma_i]^\dagger = -\frac{1}{4}[\gamma_0^\dagger,\gamma_i^\dagger]$

remember from last lecture $\gamma_0^\dagger = -\gamma_0$, $\gamma_i^\dagger = \gamma_i$

$\rightsquigarrow (S_{0i})^\dagger = S_{0i}$ and $(S_{ij})^\dagger = -S_{ij}$

therefore $\delta(\psi^\dagger\psi) \neq 0$ not a scalar :-(

fixed by $\boxed{\bar\psi = i\psi^\dagger\gamma^0}$ Dirac conjugation

because: $S_{0i}^\dagger\gamma^0 = \frac{1}{4}(\gamma_0\gamma_i - \gamma_i\gamma_0)\gamma_0$

$\gamma_0\gamma_0 = -\mathbb{1}$ $\rightarrow$ $= \frac{1}{4}\gamma_0(\gamma_i\gamma_0 - \gamma_0\gamma_i)$

$= -\gamma_0 S_{0i}$ //

The Dirac equation arises from the action

$$\boxed{S = -\int d^4x\,(\bar\psi\,\partial\!\!\!/\,\psi + m\bar\psi\psi)}$$

Variation with respect to $\bar\psi$ and ψ gives

$$\delta S = -\int d^4x\,[\delta\bar\psi\underbrace{(\partial\!\!\!/\,\psi + m\psi)}_{\text{spinor}} + (-\partial_\mu\bar\psi\gamma^\mu + m\bar\psi)\delta\psi]$$

$$\overline{\gamma^\mu\partial_\mu\psi + m\psi} = i\partial_\mu\psi^\dagger\gamma^{\mu\dagger}\gamma_0 + m\underbrace{i\psi^\dagger\gamma_0}_{\bar\psi}$$

like before

$$\gamma^{\mu\dagger}\gamma_0 = -\gamma_0\gamma^\mu \quad \left\{ \begin{array}{l} \gamma^{0\dagger}\gamma_0 = -\gamma_0\gamma^0 \\ \gamma^{i\dagger}\gamma_0 = -\gamma_0\gamma^i \end{array} \right.$$

$\rightsquigarrow$ $= -\partial_\mu\bar\psi\gamma^\mu + m\bar\psi$

the action reproduces the Dirac equation :-)

10.4. Conserved charges

S is by construction invariant under Poincaré Symm.

$\rightsquigarrow$ compute the conserved current wrt. time shifts

$$j^{\mu} = \frac{1}{2}\left(\bar{\Psi}\gamma^{\mu}\dot{\Psi} - \dot{\bar{\Psi}}\gamma^{\mu}\Psi\right)$$

Noether's theorem $\quad j^0 = \rho =$ energy density

$$\rho = \frac{i}{2}\left(\dot{\Psi}^{\dagger}\Psi - \Psi^{\dagger}\dot{\Psi}\right)$$

check for a simple solution:

$\partial_i \Psi = 0 \quad \rightsquigarrow \quad \Psi = \Psi(t)$ reduces Dirac equation

to $\quad \gamma^0 \dot{\Psi} + m\Psi = 0 \quad$ for $\Psi = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \\ d(t) \end{pmatrix}$

we then get
$$\left.\begin{aligned} -i\dot{a} + ma &= 0 \\ -i\dot{b} + mb &= 0 \\ i\dot{c} + mc &= 0 \\ i\dot{d} + md &= 0 \end{aligned}\right\} \text{ solved by } \quad \Psi = \begin{pmatrix} a_0 e^{-imt} \\ b_0 e^{-imt} \\ c_0 e^{imt} \\ d_0 e^{imt} \end{pmatrix}$$

with energy density $\quad \rho = m\Big(|a_0|^2 + |b_0|^2 \underbrace{- |c_0|^2 - |d_0|^2}_{\text{negative energy states}}\Big)$

$\rightsquigarrow$ resolution Dirac sea in QM