10. Spontaneous Symmetry Breaking

remember: $\boxed{\text{Symmetries in (Q)FT}}$

global symmetries $\qquad\qquad$ local = gauge symmetries

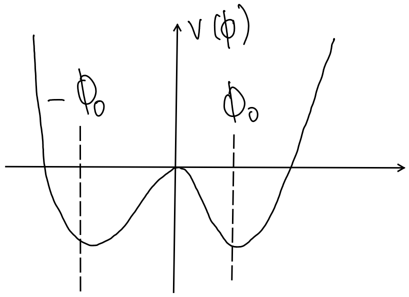
Today, we see why and how they can be broken.

10.1. Global Symmetries & Goldstone's Theorem

Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 = T - V$

introduce $m = i\mu$ in ϕ^4-theory

Theory has a $\mathbb{Z}_2$-symmetry $\phi \longrightarrow -\phi$ and the potential



$$V = -\tfrac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

$\phi_0 = \sqrt{\frac{6}{\lambda}}\,\mu$ with minimum $\pm\phi_0$

called vacuum expectation value of ϕ

expanding $\phi(x) = \phi_0 + \sigma(x)$ results in

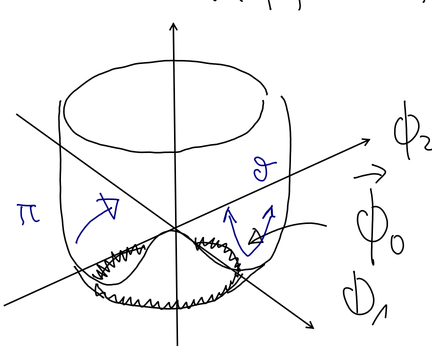
$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \sigma)^2 \underbrace{- \frac{1}{2}(2\mu^2)\sigma^2}_{\text{mass term for } m=\sqrt{2}\mu} \underbrace{- \sqrt{\frac{\lambda}{6}}\,\mu\,\sigma^3}_{\text{brakes } \mathbb{Z}_2\text{-Sym. } \phi \longrightarrow -\phi} - \frac{\lambda}{4!}\sigma^4$$

More interesting are continuous symmetries: $\quad i = 1, \dots, N$

like $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[|\phi^i|^2\right]^2 = T - V$

which has a $\phi^i \to R^i{}_j\phi^j$, $R^T R = \mathbb{1}$, $O(N)$-Symmetry

with $V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}\left[(\phi^i)^2\right]^2$ and minima at



$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}$$

$\hat{=}$ vector $\vec{\phi}_0$ with fixed length choose (hyper)-spherical coordinates such that

$$\phi^i(x) = \left(\frac{\mu}{\sqrt{\lambda}} + \sigma(x),\ \pi^k(x)\right)$$ and we hit the minima $\vec{\Phi}_0$ for $\sigma(x) = 0$

$\uparrow$ radius $\uparrow$ angles

results in:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3$$
$$- \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2$$

- σ is massive ($m_\sigma = \sqrt{2}\mu$), like before
- π^k, $k = 1, \dots, N-1$, massless fields with $O(N-1)$-sym.

The observation from this example generalises:

> For every spontaneously broken continuous symmetry, the theory must contain a massless particle

= Goldstone's theorem

Check: $\dim(O(N)) = \frac{N(N-1)}{2}$, $\dim(O(N-1)) = \frac{(N-1)(N-2)}{2}$

broken symmetry generators $= \dim(O(N)) - \dim(O(N-1)) = N-1 = \#\pi$'s

Works not only classically, but to all orders in perturbation theory

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10.2. Gauge Symmetries & The Higgs Mechanism

Simplest example: Abelian gauge theory with

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi),$$ remember $D_\mu = \partial_\mu + ieA_\mu$,

and we find the local symmetry $\phi(x) \to e^{i\alpha(x)}\phi(x)$ and
$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

motivated by the examples from the last section we choose:

$$V(\phi) = -\mu^2 \phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2$$ with minima at

$$\langle\phi\rangle = \phi_0 = \left(\frac{\mu^2}{\lambda}\right)^{1/2}$$

again we expand around the minimum: $\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\,\phi_2(x)\right)$

with $V(\phi) = -\frac{1}{2\lambda}\mu^4 + \frac{1}{2}\,2\mu^2\,\phi_1^2 + \mathcal{O}(\phi_i^3)$

$\longrightarrow$ ϕ_1 aquires mass $m = \sqrt{2}\,\mu$

ϕ_2 is the Goldstone boson, but something new happens for

Kinetic term:

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}\,e\,\phi_0\,A_\mu\partial^\mu\phi_2 + \underbrace{e^2\phi_0^2\,A_\mu A^\mu}_{\text{new mass term for the photon}} + \dots$$

new mass term for the photon $= \frac{1}{2} m_A^2\, A_\mu A^\mu$

with $m_A = \sqrt{2}\,e\,\phi_0$

Note: In contrast to the examples before, we can remove the Goldstone boson ϕ_2 by gauge fixing (unitary gauge).
On the other hand the massive $U(1)$-gauge field allows a longitudinal mode (in contrast to the massless on ↗ section 3.5).

The Goldstone boson gets "eaten up" by this mode.

Non-abelian generalisation:

gauge transformation: $\phi_i \to \left[\delta_i^{\ j} + i\,\alpha^a (t_a)_i^{\ j}\right]\phi_j$

here real fields; purely imag.

with covariant derivative: $D_\mu\phi = (\partial_\mu - i g A_\mu^a t_a)\phi$ $\qquad T_a = i t_a$

$$\text{Kinetic term: } \frac{1}{2}(D_\mu\phi_i)^2 = \frac{1}{2}(\partial_\mu\phi_i)^2 + g A_\mu^a\left(\partial^\mu\phi^i (T_a)_i^{\ j}\phi_j\right) + \frac{1}{2} g^2 A_\mu^a A^{b\mu}\,(T^a\phi)^i\,(T^b\phi)_i$$

expanding around VEV (vacuum expectation value) $\langle\phi_i\rangle = (\phi_0)_i$:

$\longrightarrow$ new mass term $\Delta\mathcal{L} = \frac{1}{2} m_{ab}^2\, A_\mu^a A^{b\mu}$ with

$$m_{ab}^2 = g^2\,(T^a\phi_0)^i\,(T^b\phi_0)_i$$

The matrix m^2_{ab} is positive semidefinite $\longrightarrow$ diagonalisable with eigenvalues $\lambda_a \geq 0$

If $\lambda_a = 0$, $(T_a)_i{}^j (\phi_0)_j = 0$ unbroken symmetry

$\longrightarrow$ no Goldstone boson & mass term for gauge boson

SU(2) example: $t_a = \frac{\sigma_a}{2}$ $\leftarrow$ Pauli matrices

$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ results in the kinetic term

$$|D_\mu \phi|^2 = \frac{1}{2} g^2 \begin{pmatrix} 0 & v \end{pmatrix} t_a t_b \begin{pmatrix} 0 \\ v \end{pmatrix} A^a_\mu A^b_\nu + \ldots$$

$\rightsquigarrow \Delta \mathcal{L} = \frac{g^2 v^2}{8} A^a_\mu A_a^\mu$ $\rightsquigarrow$ all three gauge bosons aquire the same mass

$$m_A = \frac{gv}{2}$$