12.3. Dynkin index of embedding

Take $i: h \to g$ as the map which captures the embedding of $h \subset g$ in g.

Compute: Killing form $K_{h\subset g}(x,y) := K_g(i(x), i(y))$
$x, y \in h$

and comparing with $K_h(x,y)$ gives:

$$c\, I_{h\subset g} := \frac{K_g(i(x), i(y))}{K_h(x,y)} \quad \text{with} \quad c = \frac{I_{ad}(g)}{I_{ad}(h)}$$

remember: $I_{ad}(g) / I_{ad}(h)$ are the normalisations we introduced,

$$K_g(X,Y) = \frac{1}{I_{ad}(g)} \operatorname{tr}_{ad}(ad_X \circ ad_Y),$$

in section 4.6.

note: normalisation of Killing form fixed by highest weight.

$$\leadsto \boxed{I_{h\subset g} = \frac{(\theta(g), \theta(g))}{(\theta(h), \theta(h))}}$$

highest weight in g and in h

$=$ Dynkin index of the embedding

For regular embedding: $\theta(h) \in$ roots of g

$\leadsto$
- for simply laced $I_{h\subset g} = 1$
- for B_r, C_r or F_4 $I_{h\subset g} \in \{1, 2\}$
- for G_2 $I_{h\subset g} \in \{1, 2, 3\}$.

12.4. Special subalgebras

remember: $V = n$ dim module of g

$\longrightarrow \rho_V : g \longrightarrow gl(V) \cong gl(n, \mathbb{C})$

even more:

1.) g is simple $\rho_V(X) \in sl(n)$ and

2.) V is self-conjugated $\rho_V(X) \in so(n)$ or $sp(n)$

for the adjoint

for example 1.) $\to$ $E_6 \hookrightarrow sl(27, \mathbb{C})$ and

2.) $\to$ $g \hookrightarrow so(\dim g)$

are maximal special embeddings

Semi simple embeddings

remember: $A_{n-1} \cong su(n)$ and the reducible module

$$V = V_{\square} \oplus V_{\bar{\square}} = \left.\begin{array}{c}\square\\ \square\\ \vdots\end{array}\right\} n-1 \quad \text{with}$$

$$\rho_V \in so(\dim V) = so(2n)$$

$\to$ $sl(n) \hookrightarrow so(2n)$ but not maximal

real version of $su(n)$

The corresponding maximal embedding is

$$u(1) \oplus sl(n) \hookrightarrow so(2n)$$

and for the other classical Lie algebras, we find

$$sl(m) \oplus sl(n) \hookrightarrow sl(m \cdot n),$$
$$so(m) \oplus so(n) \hookrightarrow so(m \cdot n),$$
$$sp(m) \oplus sp(n) \hookrightarrow so(4m \cdot n),$$
$$so(m) \oplus sp(n) \hookrightarrow sp(m \cdot n) \quad \text{and}$$
$$so(2m+1) \oplus so(2n+1) \hookrightarrow so(2m+2n+2).$$

For the exceptional's, things are more complicated and one finds:

$\mathfrak{g}$	maximal special subalgebras
E_6	$A_2^{[9]}$, $G_2^{[3]}$, $C_4^{[1]}$, $F_4^{[1]}$, $A_2^{[2]} \oplus G_2^{[1]}$ — Dynkin index of embedding
E_7	$A_1^{[231]}$, $A_1^{[399]}$, $A_2^{[21]}$, $A_1^{[15]} \oplus A_1^{[24]}$, $A_1^{[7]} \oplus G_2^{[2]}$, $A_1^{[3]} \oplus F_4^{[1]}$, $C_3^{[1]} \oplus G_2^{[1]}$
E_8	$A_1^{[520]}$, $A_1^{[760]}$, $A_1^{[1240]}$, $B_2^{[12]}$, $A_1^{[16]} \oplus A_2^{[6]}$, $F_4^{[1]} \otimes G_2^{[1]}$
F_4	$A_1^{[156]}$, $A_1^{[8]} \oplus G_2^{[1]}$
G_2	$A_1^{[28]}$

12.5. Branching rules

remember: In physics Lie algebras do not appear directly but through representations.

Question: How do irreps of a Lie algebra $\mathfrak{g}$ decompose into irreps of it's subalgebra $\mathfrak{h}$?

Answer: Encoded in branching rules

Take V_Λ a module of $\mathfrak{g}$ with highest weight Λ and V_{λ_i}: modules of $\mathfrak{h}$ —— " ——s λ_i

then $V_\Lambda(\mathfrak{g}) \longrightarrow \bigoplus_j V_{\lambda_j}(\mathfrak{h})$ and in terms of characters

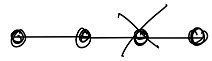
$$\chi_\Lambda^{(\mathfrak{g})} = \sum_j \chi_{\lambda_j}^{(\mathfrak{h})}$$

In particular: $\dim V_\Lambda = \sum_j \dim V_{\lambda_j}$

To obtain branching rules we need

1) projection matrix $P_{\mathfrak{g}\to\mathfrak{h}}$ relating the weight of $\mathfrak{g}$ to the weights of the subagebra $\mathfrak{h}$
2) after applying $P_{\mathfrak{g}\to\mathfrak{h}}$, we use the same technique

to find all irreps we used for the decomp. of tensor products in section 11.3.

Example: $su(\bar{5}) \overset{1}{=}$ 

$\longrightarrow su(\bar{5}) \supset su(3) \times su(2) \times u(1)$

projection matrix $P_{g \to h} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 6 & 3 \end{pmatrix}$

LieART:

ProjectionMatrix[SU5, ProductAlgebra[SU3, SU2, U1]]

$su(5)$		$su(3)$	$\oplus\ su(2)$	$\oplus\ u(1)$	
$\square$: $\boxed{1\ 0\ 0\ 0}$	$P_{g\to h}$:	$\boxed{1\ 0}$	$\boxed{0}$	2	$(3,1)_2$
$\downarrow \boxed{-1\ 1\ 0\ 0}$		$\boxed{-1\ 1}$	$\boxed{0}$	2	
$\downarrow \boxed{0\ -1\ 1\ 0}$		$\boxed{0\ -1}$	$\boxed{0}$	2	
$\downarrow \boxed{0\ 0\ -1\ 1}$		$\boxed{0\ 0}$	$\boxed{1}$	-3	$(1,2)_{-3}$
$\downarrow \boxed{0\ 0\ 0\ -1}$		$\boxed{0\ 0}$	$\boxed{-1}$	-3	

Project[$P_{g\to h}$, WeightSystem[Irrep[SU5][$\bar{5}$]]]

result: $5 \to (1,2)_{-3} + (3,1)_2$

= DecomposeIrrep[Irrep[SU$\bar{5}$][5], ProductAlgebra[...]]