11.3. Decomposition of tensor products

remember: last lecture characters with properties

$$\chi_{V_1 \oplus V_2}(\mu) = \chi_{V_1}(\mu) + \chi_{V_2}(\mu) \quad \text{and}$$

$$\chi_{V_1 \otimes V_2}(\mu) = \chi_{V_1}(\mu) \cdot \chi_{V_2}(\mu)$$

very useful to decompose tensor products

algorithm:

$$\chi_{V_1 \otimes V_2}(\mu) = \sum_{\lambda_1, \lambda_2} \mathrm{mult}_{V_1}(\lambda_1)\, \mathrm{mult}_{V_2}(\lambda_2)\, e^{(\mu, \lambda_1 + \lambda_2)}$$

⚡ lots of weights to deal with $\rightsquigarrow$ just consider dominant weights with only non-negative λ^i's

1) find the highest (with highest level) dominant weight

2) use it as highest weight and subtract the dominant weight system of the corresponding irrep

3) repeat until no weights are left

Example: su(3) $\quad 3 \otimes 3 = \bar{3} \oplus 6$

or $\quad \square \otimes \square = \begin{smallmatrix}\square \\ \square\end{smallmatrix} \oplus \square\square$

3:
$$\begin{tikzcd}
\boxed{1 \;\; 0} \arrow[d, "\alpha_1"] \\
\boxed{-1 \;\; 1} \arrow[d, "\alpha_2"] \\
\boxed{0 \;\; 1}
\end{tikzcd}$$

6:
$$\begin{tikzcd}
& \boxed{2 \;\; 0} \arrow[d, "\alpha_1"] & \\
& \boxed{0 \;\; 1} \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\
\boxed{-2 \;\; 2} \arrow[dr, "\alpha_2"'] & & \boxed{1 \;\; -1} \arrow[dl, "\alpha_1"] \\
& \boxed{-1 \;\; 0} \arrow[d, "\alpha_2"'] & \\
& \boxed{0 \;\; -2} &
\end{tikzcd}$$

dominant weights

$$3 \otimes 3 \;\hat{=}\; \underbrace{\boxed{2 \;\; 0}}_{\text{level } 2} \oplus \underbrace{2 \cdot \boxed{0 \;\; 1}}_{\text{level } 1} \oplus 2\,\boxed{1 \;\; -1} \oplus 2 \cdot \boxed{-1 \;\; 0} \oplus \boxed{0 \;\; -2}$$
$$\oplus\; \boxed{-2 \;\; 2}$$

$\longrightarrow \; 3 \otimes 3 = \bar{3} \oplus 6$

Lie ART: DecomposeProduct[Irrep[SU3][3], Irrep[SU3][3]]

12.1. Subalgebras

motivation: Lie algebra $\sim$ symmetry in physics

$\cup$ | spontaneous $\downarrow$ breaking

Lie subalgebra $\sim$ residual sym.

We already encountered two subalgebras, the Cartan and su(2).

Now: maximal subalgebras = subalgebra h that can't be found in any larger sub. alg. of g other than g itself.

- regular: all step operators (roots) of h are also step ops. of g
- special: other cases

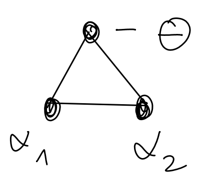
regular maximal subalgebras can be all found by employing extended Dynkin diagrams.

12.2 Extended Dynkin diagrams

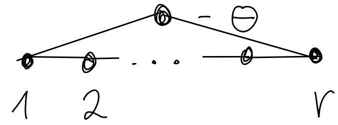
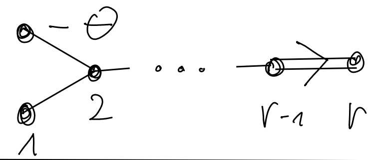
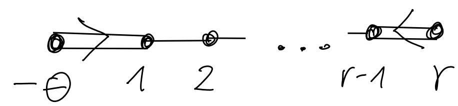
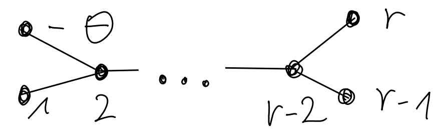
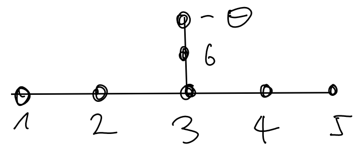
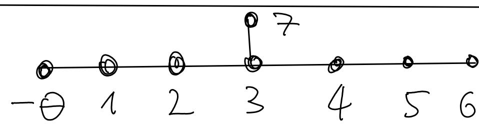
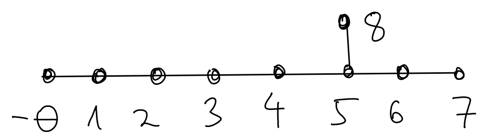
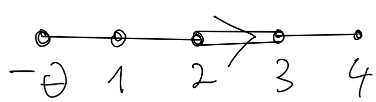
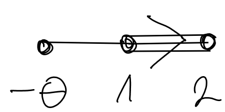
Def.: g is a simple Lie algebra with simple roots α_i and θ denotes the highest weight of the adjoint. The extended Dynkin diagram of g is obtained by adding a node corresponding to $-\theta$

Example: su(3), $\theta = \boxed{1\ 1} = \alpha_1 + \alpha_2$

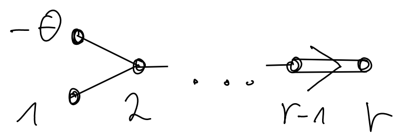
$= \boxed{2\ -1} + \boxed{-1\ 2}$

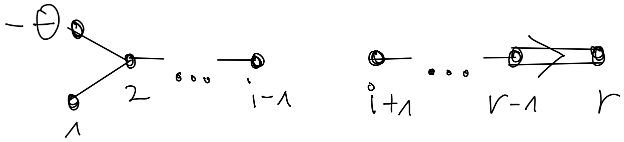


⚡ We excluded such diagrams in the classification in section 7.2. They do not describe simple Lie algebras but are only a tool here.

$\mathfrak{g}$	highest root	Extended Dynkin diagram	
A_r	$\boxed{1\ 0 \dots 0\ 1}$		$\hat{A}_r$
B_r	$\boxed{0\ 1\ 0 \dots 0}$		$\hat{B}_r$
C_r	$\boxed{2\ 0 \dots 0}$		$\hat{C}_r$
D_r	$\boxed{0\ 1\ 0 \dots 0}$		$\hat{D}_r$
E_6	$\boxed{0 \dots 0\ 1}$		$\hat{E}_6$
E_7	$\boxed{1\ 0 \dots 0}$		$\hat{E}_7$
E_8	$\boxed{1\ 0 \dots 0}$		$\hat{E}_8$
F_4	$\boxed{1\ 0 \dots 0}$		$\hat{F}_4$
G_2	$\boxed{1\ 0}$		$\hat{G}_2$

By removing a node we obtain $\mathfrak{g}$ itself or a regular (semi)simple subalgebra $\mathfrak{h}$.

Example: Take $\hat{B}_r$  and remove the i-th simple root:

 $= D_i \oplus B_{r-i},\ 1 \leq i \leq r$

or $so(2i) \oplus so(2r-2i+1) \subset so(2r+1)$ or for the Lie groups

$$SO(2i) \times SO(2r-2i+1) \subset SO(2r+1)$$

For $\hat{A}_r$ we need to remove <u>two</u> nodes, where the second node introduces an additional $U(1)$ factor.

$\longrightarrow$ regular subalgebras of A_r are

$$A_{r-1} \times U(1) \quad \text{and} \quad A_{k-1} \times U(1) \times A_{r-k} \qquad 1 < k < r$$

$$\text{or} \quad SU(k) \times U(1) \times SU(r-k+1) \subset SU(r+1)$$

$$\phantom{\text{or} \quad} \| \qquad\qquad\qquad\qquad \|$$

$$SU(m) \times U(1) \times SU(n) \subset SU(m+n)$$

What is the additional $U(1)$ generator?

$$t = \begin{pmatrix} i\alpha \, \mathbb{1}_m & 0 \\ 0 & i\beta \, \mathbb{1}_n \end{pmatrix} \qquad t^\dagger = -t \rightsquigarrow \alpha, \beta \in \mathbb{R}$$

$$\mathrm{Tr}\, t = 0 = i(m\alpha + n\beta)$$

i.e. $\alpha = \frac{m}{m+n}$ & $\beta = -\frac{n}{m+n}$