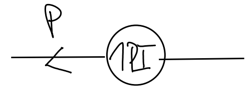
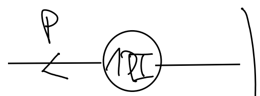
8.2. The Callan-Symanzik Equation

Problem: Cutoff Λ usually break symmetries of the theory. We know better regularisation methods, like dim. reg. $\longrightarrow$ use them! How?

Remember renormalisation conditions

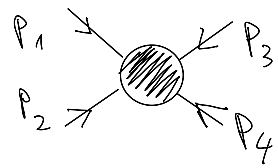


$= 0$

$\frac{d}{dp^2}\Big($  $\Big) = 0$

at $p^2 = -M^2$

different from $s = 4m^2$ and $t = u = 0$ for 7.2. but better for calc. here



$= -i\lambda$ at $(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -M^2 = s = t = u$

M is called renormalisation scale ↗ EX 11

⚡ We currently just discuss massless theories, masses later.

Question: How does a change of M affects Green's functions?

$$G^{(n)}(x_1,\dots,x_n) = \langle 0|T\,\phi(x_1)\dots\phi(x_n)|0\rangle_{\text{connected}}$$

$$M \to M + \delta M \qquad \phi \to (1+\delta\eta)\,\phi$$

$$\lambda \to \lambda + \delta\lambda$$

$$G^{(n)}(x_1,\dots,x_n) \to (1+n\,\delta\eta)\,G^{(n)}$$

Function of M and λ

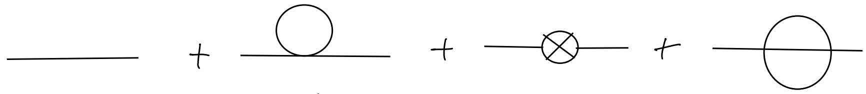
$$dG^{(n)}(x_1,\dots,x_n) = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\,\delta\eta\, G^{(n)}$$

$$\beta = \frac{M}{\delta M}\delta\lambda \qquad \gamma = -\frac{M}{\delta M}\delta\eta$$

$$\Leftrightarrow \boxed{\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] G^{(n)}(x_1,\dots,x_n;M,\lambda) = 0}$$

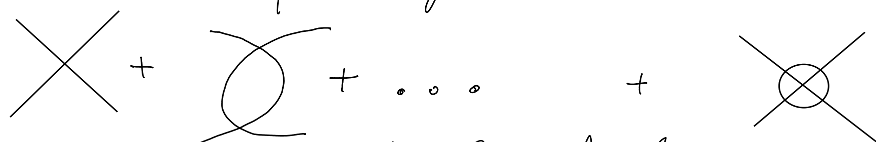
Callan-Symanzik equation

Example: ϕ^4 - theory



$$G^{(2)}(p) = \text{—} + \ldots + \ldots + \ldots + \ldots$$

remember: wave function renormalisation in ϕ^4 theory just @ two loops



$$G^{(4)} = \ldots + \ldots + \ldots + \ldots + \mathcal{O}(\lambda^3)$$

s-, t- & u-chanel

$$\ldots = -i\lambda + (-i\lambda)^2\left[iV(s) + iV(t) + iV(u)\right] - i\delta_\lambda$$

$$\delta_\lambda = (-i\lambda)^2\, 3V(-M^2) = \frac{3\lambda^2}{2(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-d/2)}{(x(1-x)M^2)^{2-d/2}}$$

renormalisation condition $s = t = u = -M^2$

$$\hookrightarrow \delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left[\frac{1}{2-d/2} - \log M^2 + \text{finit}\right] \quad \text{for } m = 0 \ \text{!}$$

We can now evaluate:

1) $$M\frac{\partial}{\partial M}G^{(4)} = M\frac{\partial}{\partial M}(-i\delta_\lambda) = \frac{3i\lambda^2}{(4\pi)^2} \quad \text{and}$$

2) $$\frac{\partial}{\partial\lambda}G^{(4)} = -i \quad \text{therefore}$$

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 4\gamma(\lambda)\right]G^{(4)} = 0 \quad \text{implies}$$

$$\beta(\lambda) - 4i\gamma(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 2\gamma(\lambda)\right]G^{(2)} = 0$$

first non-trivial contribution at two loops

$$\longrightarrow \gamma(\lambda) = 0 + \mathcal{O}(\lambda) \quad \text{and} \quad \beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

For a generic massless scalar theory:

$$G^{(2)}(p) = \text{———} + (\text{one loop diagrams}) + \text{——}\otimes\text{——} + \dots$$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A \log\frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}\left(ip^2\delta_Z\right)\frac{i}{p^2} + \dots$$

at leading order $O(\lambda)$ we can neglect $\beta(\lambda)$ and find

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M}\delta_Z = \boxed{-A = \gamma}$$

$$G^{(2)} = \begin{pmatrix}\text{tree-level}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{1 PI loop}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{Vertex}\\ \text{counterterm}\end{pmatrix} + \begin{pmatrix}\text{external leg}\\ \text{corrections}\end{pmatrix}$$

$$= \left(\prod_i \frac{i}{p_i^2}\right)\Big[-ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i\left(A_i \log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i}\right)$$

$$+ \text{finit terms}\Big]$$

invariant build from p_i's all $= -M^2$ by renormalisation condition

to lowest order we then find:

$$\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_i \delta_{Z_i}\right) = \boxed{-2B - g\sum_i A_i = \beta(g)}$$

Interpretation of $\gamma(\lambda)$ and $\beta(\lambda)$

remember: $\phi = Z(M)^{-1/2}\phi_0 \longleftarrow$ bare

renormalised

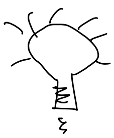
$$\delta\eta = \frac{Z(M+\delta M)^{-1/2}}{Z(M)^{-1/2}} - 1 \rightsquigarrow \gamma(\lambda) = \frac{1}{2}\frac{M}{Z}\frac{\partial}{\partial M}Z$$

Same argument for $\beta \rightsquigarrow \beta(\lambda) = M\frac{\partial}{\partial M}\lambda$

rates of change of normalisation (γ) and couplings (β) when energy scale M changes

8.3 Evolution of the Coupling Constants

Orginal motivation: Integrate out one mass shell after an other. ⇝ renormalisation group flow



Callan - Symanzik equation = differential eq.
Solve (integrate) it!

$$G^{(n)}(p,\lambda) = \hat{G}^{(n)}(\bar{\lambda}(p;\lambda)) \cdot \exp\left(- \int_{p'=M}^{p'=p} d\log\left(\frac{p'}{M}\right) \cdot n\left[1-\gamma(\bar{\lambda}(p';\lambda))\right]\right)$$

initial condition (measured at $p^2 = -M$)

$\bar{\lambda}$ = running couplings which solve:

$$\frac{d}{d\log\left(\frac{p}{M}\right)} \bar{\lambda}(p;\lambda) = \beta(\bar{\lambda}) \quad \text{and} \quad \bar{\lambda}(M;\lambda) = \lambda$$

$\beta(\bar{\lambda}) = 0$ is a fixed point of the flow