remember last lecture: scalar electrodynamics

$$\mathcal{L} = D_\mu \phi \, \overline{D^\mu \phi} - m^2 \phi \bar{\phi} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with cov. deriv. $D_\mu \phi = (\partial_\mu + ie A_\mu)\phi$

equations of motion: gauged Klein-Gordon

$$0 = \frac{\delta \mathcal{L}}{\delta \phi} = (D_\mu D^\mu - m^2)\bar{\phi}, \quad \frac{\delta \mathcal{L}}{\delta \bar{\phi}} = \boxed{(D_\mu D^\mu - m^2)\phi = 0}$$

$$0 = \frac{\delta \mathcal{L}}{\delta A_\mu} = \partial_\nu F^{\nu\mu} - J^\mu \quad \rightsquigarrow \quad \boxed{\partial_\nu F^{\nu\mu} = J^\mu}$$

inhomogeneous Maxwell eqs.

with current $J^\mu = -ie(\bar{\phi} D^\mu \phi - \phi \overline{D^\mu \phi})$ now with cov. derivatives

Question: Why no cov. deriv. for $\partial_\mu F^{\mu\nu} = J^\nu$?

$\rightsquigarrow$ remember $\delta_\Lambda F_{\mu\nu} = 2\partial_{[\mu} \delta A_{\nu]} = 2\partial_{[\mu}\left(\frac{1}{e}\partial_{\nu]}\Lambda\right) = 0$

therefore $\partial_\mu F_{\nu\rho} = D_\mu F_{\nu\rho}$

because no compensation is required or

$e(\text{photon}) = 0$; the photon is not charged

8. Non-Abelian gauge theories

so far: $\phi' = e^{-i\Lambda_1}\phi =: T_{\Lambda_1}\phi, \quad \phi'' = e^{-i\Lambda_2}\phi' = T_{\Lambda_2}\phi'$

$$\phi'' = T_{\Lambda_2} T_{\Lambda_1} \phi = e^{-i(\Lambda_2 + \Lambda_1)}\phi = T_{\Lambda_1} T_{\Lambda_2}\phi$$

it does not matter which transformation is done first

$\rightsquigarrow$ Abelian group ($U(1)$) with $T_{\Lambda_1} T_{\Lambda_2}\phi = T_{\Lambda_2} T_{\Lambda_1}\phi$

BUT in general $T_{\Lambda_1} T_{\Lambda_2} \circ \neq T_{\Lambda_2} T_{\Lambda_1} \circ$ (non-Abelian group)

<u>Example:</u> SU(2) we encountered in sec. 4.4

i.e. two complex KG fields $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

which transform as

$$\Phi' = T_{\vec{\alpha}} \Phi = \exp\left(i \frac{\alpha^i \sigma_i}{2}\right) \Phi$$

2x2 Pauli-matrices with $i = 1, 2, 3$

<u>infinitesimal:</u> $T_{\vec{\alpha}} = \mathbb{1} + i \frac{\alpha^i \sigma_i}{2} + \frac{1}{2}\left(\frac{i}{2}\right)^2 \alpha^i \sigma_i \alpha^j \sigma_j + \dots$

$$T_{\vec{\alpha}} \Phi = \Phi + \delta_{\vec{\alpha}} \Phi + \mathcal{O}(\alpha^2) \rightsquigarrow \delta_{\vec{\alpha}} \Phi = i \frac{\alpha^i \sigma_i}{\alpha} \Phi$$

therefore $\left(T_{\vec{\alpha}_1} T_{\vec{\alpha}_2} - T_{\vec{\alpha}_2} T_{\vec{\alpha}_1}\right) \Phi = \alpha_1^i \alpha_2^j \left[i \frac{\sigma_i}{2}, i \frac{\sigma_j}{2} \right] + \dots$

remember $\boxed{[\sigma_i, \sigma_j] = 2 i \varepsilon_{ij}{}^{k} \sigma_k}$ Lie algebra su(2)

<u>Lagrangian:</u> defining $\bar{\Phi} = \Phi^\dagger = (\bar{\phi}_1 \ \bar{\phi}_2)$, we write

$$\mathcal{L} = \partial_\mu \bar{\Phi} \partial^\mu \Phi - m^2 \bar{\Phi} \Phi = (\partial_\mu \bar{\phi}_1 \ \ \partial_\mu \bar{\phi}_2) \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix}$$

$$- m^2 (\bar{\phi}_1 \ \ \bar{\phi}_2) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \partial_\mu \bar{\phi}_1 \partial^\mu \phi_1 + \partial_\mu \bar{\phi}_2 \partial^\mu \phi_2$$

$- m^2 \bar{\phi}_1 \phi_1 - m^2 \bar{\phi}_2 \phi_2 \rightsquigarrow$ 2 complex KG fields

<u>check</u> invariance: $\left(T_{\vec{\alpha}} \Phi\right)^\dagger = \left(\Phi'\right)^\dagger = \Phi^\dagger \left(T_{\vec{\alpha}}\right)^\dagger$

with $\sigma_i^\dagger = \sigma_i$ we find $\left(T_{\vec{\alpha}}\right)^\dagger = \left(T_{\vec{\alpha}}\right)^{-1}$ and therefore

$$\bar{\Phi}' \Phi' = \bar{\Phi} \left(T_{\vec{\alpha}}\right)^{-1} T_{\vec{\alpha}} \Phi = \bar{\Phi} \Phi \ \checkmark$$

like for SED $\mathcal{L}$ is invariant under <u>global SU(2)</u>

7.1. Conserved currents

remember: $$\boxed{J^\mu_i = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi)} \delta \Phi(\alpha_i) + \delta \overline{\Phi}(\alpha_i) \frac{\delta \mathcal{L}}{\delta(\overline{\partial_\mu \Phi})}}$$

$$J^\mu_i = \frac{i}{2}\left(\overline{\partial_\mu \Phi}\, \sigma_i \Phi - \overline{\Phi}\, \sigma_i \partial^\mu \Phi\right)$$ compared with

$$j^\mu = -i\left(\overline{\partial_\mu \phi}\, \phi - \overline{\phi}\, \partial_\mu \phi\right)$$

still we have $\partial_\mu J^\mu_i = 0$ on-shell with conserved charges $Q_i = \int d^3x\, J^0_i$

7.2. The Yang-Mills Lagrangian

Let's gauge this global symmetry

$\rightsquigarrow$ we need covariant derivative!

electric charge

gauge field

for $U(1)$ we had: $D_\mu \phi = (\partial_\mu + ie A_\mu)\phi$

now we use $$\boxed{D_\mu \Phi = \left(\partial_\mu - ig A^i_\mu \frac{\sigma_i}{2}\right)\Phi}$$

gauge coupling

again we check:

$$\delta(D_\mu \Phi) = \partial_\mu \delta\Phi - ig\, \delta A^i_\mu \frac{\sigma_i}{2}\Phi - ig A^i_\mu \frac{\sigma_i}{2}\delta\Phi$$

$$= i\alpha^i \frac{\sigma_i}{2} D_\mu \Phi \underbrace{- \tfrac{1}{4} g\alpha^i A^j_\mu [\sigma_i, \sigma_j]\Phi + i\partial_\mu \alpha^i \frac{\sigma_i}{2}\Phi - ig\, \delta A^i_\mu \frac{\sigma_i}{2}\Phi}_{\overset{!}{=}0}$$

$\leadsto \delta A_\mu^i \frac{\sigma_i}{2} = \frac{1}{g} \partial_\mu \alpha^i \frac{\sigma_i}{2} + \frac{i}{4} g \alpha^i A_\mu^j [\sigma_i, \sigma_j]$ //

gauge transformation of gauge field

Field strength: $[D_\mu, D_\nu] \Phi =: -ig F_{\mu\nu}^i \frac{\sigma_i}{2} \Phi$

$\ldots \quad F_{\mu\nu}^i \frac{\sigma_i}{2} = 2 \partial_{[\mu} A_{\nu]}^i \frac{\sigma_i}{2} - ig A_\mu^i A_\nu^j \left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right]$

Simplifies by using $t_i = \frac{\sigma_i}{2}$ and

structure coefficients of Lie algebra

$$\boxed{[t_i, t_j] = i f_{ij}{}^k t_k}$$

results in:
$$\boxed{\begin{aligned} \delta A_\mu^i &= g^{-1} \partial_\mu \alpha^i - g \alpha^j A_\mu^k f_{jk}{}^i \\ F_{\mu\nu}^i &= 2 \partial_{[\mu} A_{\nu]}^i + g A_\mu^j A_\nu^k f_{jk}{}^i \end{aligned}}$$

remarks: - holds not just for su(2) but any Lie group
- for $f_{ij}{}^k = 0 \leadsto$ Abelian results from last lecture

Largangian: should not transform $\rightarrow$ trivial representation

Killing metric: $2 \operatorname{Tr}(t_i t_j) = \kappa_{ij}$

$2 \operatorname{Tr}\left(\frac{\sigma_i}{2} \cdot \frac{\sigma_j}{2}\right) = \frac{1}{2} \operatorname{Tr}(\sigma_i \sigma_j) = \delta_{ij} = \kappa_{ij}$

$\kappa^{ij} \hat{=}$ inverse with $\kappa^{ik} \kappa_{kj} = \delta^i_j$ (used to raise & lower Lie algebra indices) i.e. $F_{i\mu\nu} = k_{ij} F^j_{\mu\nu}$

$$\boxed{\mathcal{L} = \overline{D_\mu \Phi} D^\mu \Phi - m^2 \overline{\Phi} \Phi - \frac{1}{4} F_{i\mu\nu} F^{i\mu\nu}}$$

Scalar electrodynamics (SED)