

8. The Renormalisation Group

Last lectures: Renormalisation $\hat{=}$ absorb UV divergences into parameters of bare Lagrangian.

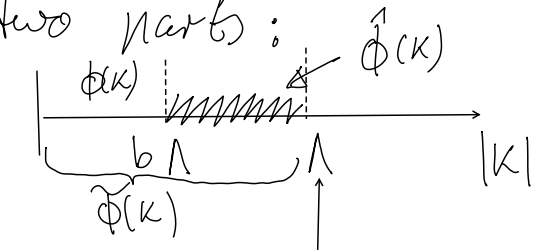
Question: Is there a better physical motivation than just "make observables finite"?

Idea (by Kenneth Wilson): Split the path integral

i.e. for $\tilde{\Phi}^4$ theory into two parts:

$$\tilde{\Phi}(k) \rightarrow \phi(k) + \hat{\Phi}(k)$$

$$\hat{\Phi}(k) = \begin{cases} \tilde{\Phi}(k) & \text{for } b\Lambda < |k| < \Lambda \\ 0 & \text{otherwise} \end{cases}$$



cutoff to regularise loops

to compute:

$$Z = \int \mathcal{D}\phi \int \mathcal{D}\hat{\Phi} \exp[-L_E(\phi + \hat{\Phi})] \quad \text{Wick-rotated Lagrangian, i.e.}$$

$$L_E(\phi) = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

$$Z = \int \mathcal{D}\phi e^{-L_E(\phi)} \int \mathcal{D}\hat{\Phi} \exp\left(-\int d^d x \left[\frac{1}{2} (\partial_\mu \hat{\Phi})^2 + \frac{1}{2} m^2 \hat{\Phi}^2 + \lambda \left(\frac{1}{6} \phi^3 \hat{\Phi} + \frac{1}{4} \phi^2 \hat{\Phi}^2 + \frac{1}{6} \phi \hat{\Phi}^3 + \frac{1}{4!} \hat{\Phi}^4 \right) \right]\right)$$

tactic: integrate out momentum $\hat{\Phi}$ to obtain

$$= \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(-\int d^d x L_{\text{eff}}(\phi)\right)$$

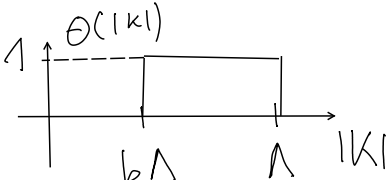
momentum \rightarrow cutoff

\rightarrow use Feynman diagrams and treat terms as perturbation ($m^2 \hat{\Phi}^2$ too, because $m^2 \ll \Lambda^2$)

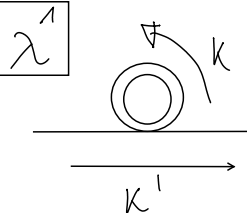
propagator:

$$\overline{\overline{\xrightarrow{k}}} \circ = \frac{\int \mathcal{D}\hat{\Phi} e^{-S_0} \hat{\Phi}(k) \hat{\Phi}(p)}{\int \mathcal{D}\hat{\Phi} e^{-S_0}} = \frac{(2\pi)^d \delta^{(d)}(k+p) \Theta(k)}{k^2}$$

with $\int \mathcal{L}_0 = \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} k^2 |\phi(k)|^2$ and $\Theta(|k|)$



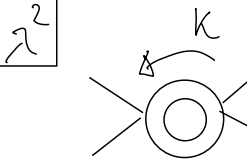
λ^1



$$= -\frac{1}{2} \int \frac{d^d k'}{(2\pi)^d} \mu |\phi(k')|^2 = -\frac{1}{2} \int d^d x \mu \phi^2$$

$$\mu = \frac{\lambda}{2} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{\lambda}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1-b^{d-2}}{d-2} \Lambda^{d-2}$$

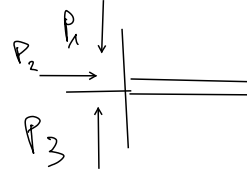
λ^2



$$= -\frac{1}{4!} \int d^d x \xi \phi^4$$

$$\xi = -\frac{3}{2} \lambda^2 \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^2 = \dots = -\frac{3\lambda^2}{16\pi^2} \log \frac{1}{b}$$

$d \rightarrow 4$



$$\propto \frac{\lambda^2}{(p_1 + p_2 + p_3)^2} \Theta(|p_1 + p_2 + p_3|)$$

and many more ...

$$\longrightarrow \mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda' \phi^4 + (\text{sum of connected diagrams})$$

Todo: clarify relation between the initial action S_E and the effective action S_{eff}

$$\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x \left[\frac{1}{2} (1 + \Delta Z) (\partial_\mu \phi)^2 + \frac{1}{2} (m^2 + \Delta m^2) \phi^2 + \frac{1}{4!} (\lambda + \Delta \lambda) \phi^4 + \Delta C (\partial_\mu \phi)^4 + \Delta D \phi^6 + \dots \right]$$

rescale also $k' = k/b$ and $x' = x \cdot b$

$$= \int d^d x' b^{-d} \left[\frac{1}{2} (1 + \Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2} (m^2 + \Delta m^2) \phi^2 + \frac{1}{4} (\lambda + \Delta \lambda) \phi^4 + \Delta C b^4 (\partial'_\mu \phi)^4 + \Delta D \phi^6 + \dots \right]$$

to match with \mathcal{L}_{eff} above we have to identify

$$\left. \begin{aligned} \phi' &= [b^{2-d} (1 + \Delta Z)]^{1/2} \phi, \\ m'^2 &= (m^2 + \Delta m^2) (1 + \Delta Z)^{-1} b^{-2}, \\ \lambda' &= (\lambda + \Delta \lambda) (1 + \Delta Z)^{-2} b^{d-4}, \\ C' &= (C + \Delta C) (1 + \Delta Z)^{-2} b^d, \text{ and} \\ D' &= (D + \Delta D) (1 + \Delta Z)^{-3} b^{2d-6} \end{aligned} \right\} \begin{array}{l} \Delta Z, \Delta m^2, \dots \\ \text{come from the} \\ \text{diagrams we} \\ \text{just computed} \end{array}$$

We are back at the point where we started and can now integrate out another momentum shell. For $b \rightarrow 1$ the transformation becomes continuous and is called renormalisation group flow.

8.1. Renormalisation Group Flow

Example: $\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \phi)^2$ is left unchanged by the flow because $\Delta m^2, \Delta \lambda, \dots$ all vanish.

→ fixed point. Close to it, we find

$$m'^2 = m^2 b^{-(2)}, \quad \lambda' = \lambda b^{(d-4)}, \quad C' = C b^{(d)}, \quad D' = D b^{(2d-6)}, \dots$$

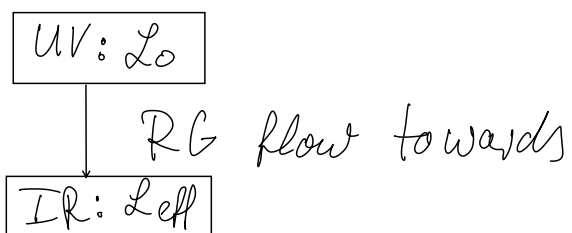
$$\begin{aligned} & -N(d/2 - 1) - M + d \quad \text{for } N: \text{ powers of } \phi \\ & = d - \underbrace{d_{N,M}}_{\text{mass dimension}} \end{aligned}$$

because $b < 1$ operators are

$$\left. \begin{aligned} d_{N,M} < d &: \text{relevant} \\ d_{N,M} > d &: \text{irrelevant} \\ d_{N,M} = 0 &: \text{marginal} \end{aligned} \right\} \begin{array}{l} \text{same as the analysis} \\ \text{of superficial degree of} \\ \text{divergence in section 7.1} \end{array}$$

$\Delta m^2, \Delta \lambda, \dots$ can change the flow

i.e. in $d=4$ $\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \log(1/b)$



→ marginally irrelevant