

10. Highest weight representations

remember: last lecture all irreps of $\text{su}(N)$ with Young tableaux

Question: What about the other simple Lie algebras?
→ today

10.1. Highest weight of $\text{su}(2)$

three generators: H, E_+, E_- ↗ negative root
certain generator ↗ positive root

$$[H, E_{\pm}] = \pm 2E_{\pm}, [E_+, E_-] = H \quad (\text{see 6.3.})$$

Irreps are characterised by highest weight vectors:

v_{λ} with

$$\begin{aligned} H v_{\lambda} &= \lambda v_{\lambda} \\ E_+ v_{\lambda} &= 0 \end{aligned}$$

check: $H E_- v_{\lambda} = E_- H v_{\lambda} - 2 E_- v_{\lambda} = (\lambda - 2) E_- v_{\lambda}$

$$\begin{array}{l} \boxed{v_{\lambda}} \\ \downarrow E_- \\ \boxed{v_{\lambda-2}} \\ \vdots \\ \boxed{v_{\lambda-2n}} \end{array} \quad \begin{aligned} v_{\lambda-2n} &= E_-^n v_{\lambda} \\ E_+ v_{\lambda-2n} &= E_+ E_- v_{\lambda-2n+2} \\ &= ([E_+, E_-] + E_- E_+) v_{\lambda-2n+2} \\ &= (H + \underbrace{E_- E_+}_{\sim}) v_{\lambda-2n+2} \\ &\sim v_{\lambda-2n+2} \end{aligned}$$

$$\left. \begin{aligned} E_- E_+ v_{\lambda-2n} &= r_n v_{\lambda-2n}, \text{ therefore} \\ E_- E_+ v_{\lambda-2n+2} &= r_{n-1} v_{\lambda-2n+2} \text{ and} \\ H v_{\lambda-2n+2} &= (\lambda - 2n + 2) v_{\lambda-2n+2} \\ r_n v_{\lambda-2n} &= (\lambda - 2n + 2 + r_{n-1}) v_{\lambda-2n} \end{aligned} \right\} \begin{aligned} E_- &\text{ on} \\ &\text{both sides} \end{aligned}$$

$$r_n = \lambda - 2n + 2 + r_{n-1} \quad \text{with} \quad r_0 = 0 \quad (E_+ v_{\lambda} = 0)$$

Solve with ansatz $r_n = \alpha n^2 + \beta n$

$\lambda = -1, \beta = \Lambda + 1 : r_n = n(\Lambda + 1 - n)$ and

$$\rightarrow E_- V_{\Lambda-2\Lambda} = 0 \quad r_{\Lambda+1} = 0$$

lowest weight vector

unitary representations: $E_+^+ = E_-^-$

↑ angular momentum in QM:

$H \sim L_z, E_+ \sim L_+, E_- \sim L_-$ and we also have

$$\vec{L}^2 = \frac{1}{4} H(H+2) + E_- E_+, \quad \vec{L}^2 v_{\Lambda} = \frac{1}{4} \Lambda(\Lambda+2) v_{\Lambda}$$

$\stackrel{\text{= spin } j = 1/2 \Lambda}{\rightarrow}$

10.2. Highest weight modules

for $su(2)$ we found $V_{(\Lambda)} = \bigoplus_{\lambda=-\Lambda}^{\Lambda} \{x_{\lambda} v_{\lambda} \mid x_{\lambda} \in \mathbb{C}\}$

labeled by Cartan generator's eigenvalues

generalisation: $V = \bigoplus_{\lambda \in \Lambda} V_{\lambda}$ with $v_{\lambda} \in V_{\lambda}$:

$$s(H_i)v_{\lambda} = \lambda^i v_{\lambda}$$

→ weight vector $\lambda = (\lambda^1, \dots, \lambda^r)$, $r = \text{rank } g$

The collection of all weights is called the weight system.

It extends the root system, which only contains the weights of the adjoint module.

↪ Not all V_{λ} 's for generic representations
↪ are one-dimensional.

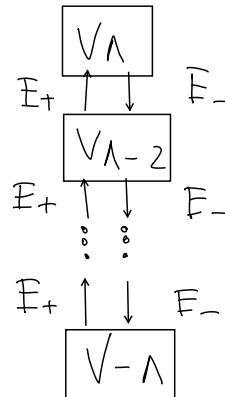
same normalisation as for roots in 6.1.

$$\lambda^i = \lambda(H_i) = (\alpha_i^V, \lambda) \in \mathbb{Z}$$

simple coroots with dual fundamental weights Λ^i

$$(\Lambda_i, \alpha_j^V) = \delta_{ij}$$

→ any weight can be written as



$$\lambda = (\lambda^i) \wedge_i \text{Dynkin labels}$$

Generalisation of su(2) modules:

For any finite dimensional, irreducible module of \mathfrak{g} there is a highest weight such that

$$s(E_\alpha) v_\lambda = 0 \quad \forall \alpha > 0 \text{ (positive roots)}$$

where v_λ is one-dimensional.

- all other weights are obtained by acting with neg. root (lowering operators)

$$\begin{aligned} s(H_i) s(E_{-\alpha}) v_\lambda &= s(H_i E_{-\alpha}) v_\lambda = s([H_i, E_{-\alpha}] + \\ E_{-\alpha} H_i) v_\lambda &= [-\alpha^i s(E_{-\alpha}) + \lambda^i s(E_{-\alpha})] v_\lambda \\ &= (\lambda - \alpha)^i s(E_{-\alpha}) v_\lambda \in V_{\lambda - \alpha} \end{aligned}$$

→ any weight can be written as $\lambda = \Lambda - \beta$,

$$\beta = \sum_i n^i \alpha_i, \quad n^i \in \mathbb{N}$$

- $\sum_i n^i$ = level or depth of the weight
- If a weight λ appears n -times in this process, it has multiplicity n .

$$\text{mult}_\Lambda(\lambda) = n \quad \rightsquigarrow \dim_{\mathbb{C}}(V_\lambda) = n$$

$$\text{mult}_\Lambda(\lambda) = 2 \sum_{\alpha > 0} \sum_{m > 0} (\lambda + m\alpha, \alpha) \text{mult}_\Lambda(\lambda + m\alpha)$$

Frenzen fal

$$\frac{\text{reduction formula}}{(1+\beta, 1+\beta) - (\lambda+\beta, \lambda+\beta)} \beta = \frac{1}{2} \sum_{\alpha > 0} \alpha$$

- for $\lambda^i > 0$, we can subtract λ^i -times the root α_i . We stop, when no further λ^i 's can be subtracted.
- The highest weight of the conjugate module \bar{V} is

$\Lambda_{\bar{V}} = -\lambda_{\min}$, λ_{\min} is the lowest weight of V
 $\rightarrow V$ is called self conjugate if $\Lambda = -\lambda_{\min}$

10.3 Example $su(3)$

Remember : $\alpha_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ see 4.7 or

$$\alpha_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{or } A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Lambda_{(1)} = \begin{matrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ \downarrow -\alpha_1 \\ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \downarrow -\alpha_2 \end{matrix}$$

$$\Lambda_{(2)} = \begin{matrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\ \downarrow -\alpha_2 \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \downarrow -\alpha_1 \end{matrix}$$

$$\begin{matrix} \Lambda_{(3)} = \begin{matrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \\ \downarrow -\alpha_1 \\ \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \\ \downarrow -\alpha_2 \\ \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix} \\ \downarrow -\alpha_1 \\ \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \\ \downarrow -\alpha_2 \end{matrix} \\ \stackrel{\cong}{=} 3 \text{ or } \square \quad \stackrel{\cong}{=} \bar{3} \text{ or } \begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} \\ \stackrel{\cong}{=} 8 = 6 + 2 \text{ or } \begin{smallmatrix} & & 1 \\ & 1 & \end{smallmatrix} \end{matrix}$$