11. Compactification

Motivation: Bosonic String ~~Weyl anomaly~~ → D = 26

Super String — " — D = 10

(Super String ← bosons + fermions)

↯ We "see" only D = 4. What to do with the rest?

→ Compactifications

space-time = $\mathbb{R}^{1,3} \times \mathcal{M}_6$ ← for superstring

($\mathcal{M}_6$: very small compact dimensions)

$\neq \mathbb{R}^{1,d}$ we studies until now

11.1. Effective action

most general Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} \left[h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) + \alpha' \phi(X) R(h) \right]$$

$$\varepsilon^{\alpha\beta} = \frac{1}{\sqrt{h}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

($G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\phi(X)$: couplings)

couplings ~ massless excitations of closed string in $\mathbb{R}^{1,25}$

non-linear σ-model (← S_P)

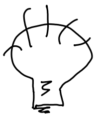
classical Weyl-invariance might be broken by quantum corrections

Weyl-anomaly

$$2\alpha' T^\alpha_\alpha = \alpha' \beta^\phi R(h) + \left(\beta^G_{\mu\nu} h^{\alpha\beta} + \beta^B_{\mu\nu} \varepsilon^{\alpha\beta} \right) \partial_\alpha X^\mu \partial_\beta X^\nu$$

we need $T^\alpha_\alpha = 0 \rightsquigarrow \beta^\phi = 0$ and $\beta^{G,B}_{\mu\nu} = 0$

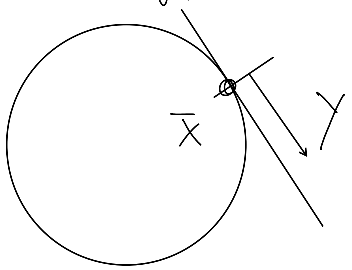
@ one-loop set $\phi = 0$ and $B_{\mu\nu} = 0$ to keep things simple



Expand $G_{\mu\nu}$ around Minkowski space

$$G_{\mu\nu} = \eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\rho\nu\lambda}(\bar{X})\, y^\rho y^\lambda + \mathcal{O}(y)$$

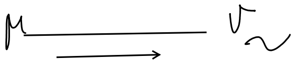
Riemann normal coordinates



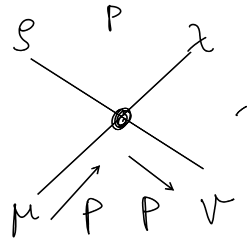
$$X^\mu = \bar{X}^\mu + \sqrt{\alpha'}\, y^\mu$$

$$S_p = \frac{1}{4\pi} \int d^2\sigma \left[\eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\rho\nu\lambda}\, y^\rho y^\lambda \right] \partial_\alpha y^\mu \partial^\alpha y^\nu$$

Feynman rules:

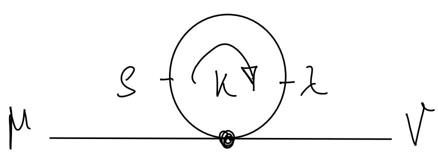
propagator 

$\sim \frac{\eta^{\mu\nu}}{p^2}$ massless-scalar

interaction 

$\sim R_{\mu\rho\nu\lambda}\, p^\mu \cdot p^\nu$ world-sheet momentum, no sum!

1 divergent diagram @ one-loop QFT 6.3



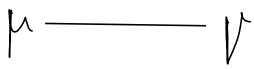
$$\sim R_{\mu\rho\nu\lambda}\, p^\mu \cdot p^\nu \int \frac{d^d k}{(2\pi)^d}\, \eta^{\rho\lambda}$$

dimensional regularisation $d = 2 - \varepsilon$

$$\sim \frac{R_{\mu\nu}\, p^\mu \cdot p^\nu}{\varepsilon} + \text{finite}$$

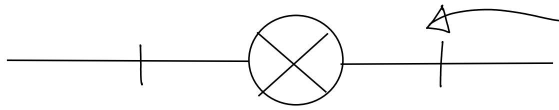
counter term QFT 7.2

wave function renormalisation

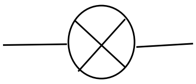


$$\sim \frac{\eta^{\mu\nu}}{p^\lambda(\eta_{\lambda\rho} + \delta_{\lambda\rho})p^\rho} \approx \frac{\eta^{\mu\nu}}{p^2} +$$

$$- \frac{\eta^{\mu\nu}}{p^2} \left(\delta_{\lambda\rho} \, p^\lambda \cdot p^\rho \right) \frac{\eta^{\mu\nu}}{p^2}$$



counter term

 $\sim - \delta_{\lambda\rho} \, p^\lambda \cdot p^\rho$

divergence is cancled if $\delta_{\mu\nu} = \frac{R_{\mu\nu}}{\varepsilon}$

(cancled → adsorbed in the counter term)

β-functions

describe how couplings change with energy scale

$$\frac{\partial G_{\mu\nu}}{\partial \log \mu} = \beta_{\mu\nu} \;\hat{=}\; @ \text{ one-loop } \# \cdot \frac{1}{\varepsilon} \text{ in counter term}$$

(μ ← energy scale)

QFT 8.2

$$\boxed{\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} = 0}$$ to kill Weyl-anomaly

($R_{\mu\nu} = 0$ ← Einstein's field equations for action)

$$S_{eff} = \int d^D x \sqrt{-G} \, R$$

> closed string theory does not only have graviton in spectrum but reproduces its dynamic

with B-field and dilaton @ one-loop (covariant deriv. ∇_ν)

$$\beta^G_{\mu\nu} = \alpha' \left(R_{\mu\nu} - \tfrac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} + 2 \nabla_\mu \nabla_\nu \phi \right) + \mathcal{O}(\alpha'^2)$$

$$\beta^B_{\mu\nu} = \alpha' \left(-\tfrac{1}{2} \nabla_\lambda H^\lambda{}_{\mu\nu} + H^\lambda{}_{\mu\nu} \nabla_\lambda \phi \right) + \mathcal{O}(\alpha'^2)$$

$$\beta^\phi = \frac{1}{4}(D - 26) + \alpha' \left[(\nabla \phi)^2 - \frac{1}{2} \nabla^2 \phi - \frac{1}{24} H^2 \right] + \mathcal{O}(\alpha'^2)$$

$\beta^\phi = 0$ and $\beta^{G,B}_{\mu\nu} = 0$ are field equations of

$$S_{eff} = \int d^D x \sqrt{-G}\, e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{(D-26)}{\alpha'} \right]$$

$H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$

Same result as from scattering massless strings and matching with effective action

Remarks ① Symmetries:

a) local

diffeomorphism $\delta G, H, \phi = L_\xi G, H, \phi$

gauge transformations $\delta B_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$

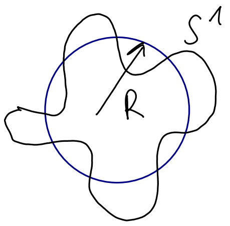
b) global $O(D,D)$ very complicated ⇝ Buscher rules ⇝ T-duality

② S_{eff} get correction is both α' + g_s but very hard to compute

11.2 Kaluza-Klein theory

Simplest possible compactification

$M = \mathbb{R}^{1,d-1} \times S^1$ ← on compact dimension



QM on ring, spectrum: $E_n \sim \left(\frac{n}{R}\right)^2$

If $R \ll 1$, $n \geq 1$ needs too much energy for us to be observed

→ at low energies wave function constant

$$G_{\mu\nu} = \begin{pmatrix} k^2 & k^2 A_j \\ k^2 A_i & g_{ij} + k^2 A_i A_j \end{pmatrix}$$

nothing depends on S^1 coordinate