7. Minimal coupling

We know global & local (gauge) symmetries.

Question: Can they be related?

7.1. Complex Klein-Gordon field

Idea: 2 real scalar fields ϕ_1 & ϕ_2

$\rightsquigarrow$ 1 complex —"— ϕ

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \bar{\phi}_{1/2} = \phi_{1/2}$$

$$\bar{\phi} = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$\mathcal{L}(\phi) = \partial_\mu \phi \partial^\mu \bar{\phi} - m^2 \phi\bar{\phi} \qquad \phi\bar{\phi} = \frac{1}{2}\left[\phi_1^2 - (i\phi_2)^2\right] = \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_2^2$$

$$= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 - m^2\phi_1^2 - m^2\phi_2^2\right)$$

$$= 1/2\left[\mathcal{L}_{KG}(\phi_1) + \mathcal{L}_{KG}(\phi_2)\right]$$

Field equations: ↯ treat ϕ and $\bar{\phi}$ as independent fields!

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \Rightarrow \partial_\mu\partial^\mu\bar{\phi} + m^2\bar{\phi} = 0 \qquad (1)$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\phi}} = 0 \Rightarrow \partial_\mu\partial^\mu\phi + m^2\phi = 0 \qquad (2)$$

$$(1) \pm (2) \Rightarrow \partial_\mu\partial^\mu\phi_{1/2} + m^2\phi_{1/2} = 0 \qquad \text{:-)}$$

Symmetries:
- Poincaré inherited from KG
- U(1) symmetry $\phi' = e^{-i\Lambda}\phi$

↯ Λ is a real constant $\bar{\phi}' = e^{i\Lambda}\bar{\phi}$

} global symmetries

The Lie group $U(1)$

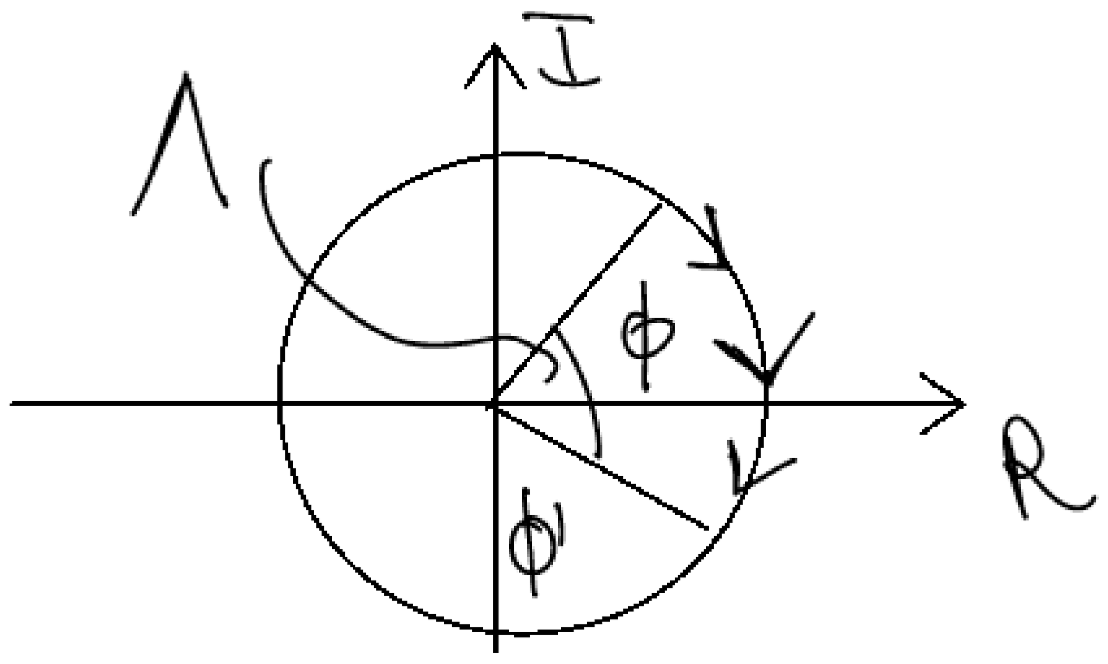
$\hat{=}$ rotation in 2 dimensions

$$\phi = \phi_R + i\phi_I$$

$$e^{-i\Lambda} = \cos\Lambda - i\sin\Lambda$$

$$\phi' = \cos\Lambda \cdot \phi_R + \sin\Lambda \cdot \phi_I + i(-\sin\Lambda \cdot \phi_R + \cos\Lambda\, \phi_I)$$

$$= \phi'_R + i\phi'_I \quad \text{or} \quad \begin{pmatrix} \phi'_R \\ \phi'_I \end{pmatrix} = \begin{pmatrix} \cos\Lambda & \sin\Lambda \\ -\sin\Lambda & \cos\Lambda \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}$$



$\leadsto$ Same as $SO(2) \sim 2$ real dim.

$U(1) \sim 1$ complex

infinitesimal version $\hat{=}$ Lie algebra $u(1)$

7.2. Conserved currents and charges

$$\phi' = \underbrace{e^{-i\Lambda}\phi\big|_{\Lambda=0}}_{\phi} + \underbrace{\Lambda\frac{\partial}{\partial\Lambda}e^{-i\Lambda}\phi\big|_{\Lambda=0}}_{-i\Lambda\phi} + \mathcal{O}(\Lambda^2) = \phi + \delta_\Lambda\phi$$

$\leadsto$ $\delta_\Lambda\phi = -i\phi\Lambda$ and therefore $\delta_\Lambda\bar{\phi} = i\bar{\phi}\Lambda$

$$\delta_\Lambda S = 0 = \int d^4x \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta_\Lambda\phi) + \frac{\delta\mathcal{L}}{\delta\phi}\delta_\Lambda\phi + \dots \delta_\Lambda\bar{\phi}\dots\right]$$

$$\overset{\text{i.b.p.}}{=} \int d^4x \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\Lambda\phi + \dots \delta_\Lambda\bar{\phi}\dots\right| -$$

$$\int d^4x \left[\underbrace{\left(\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - \frac{\delta\mathcal{L}}{\delta\phi}\right)}_{\text{field equation for } \bar{\phi} = 0 \text{ (on-shell)}}\delta_\Lambda\phi + \dots \delta_\Lambda\bar{\phi}\right] = 0$$

$\leadsto$ $\delta S = 0 = dx\, \partial_\mu j^\mu$ with

$$j^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta_\Lambda\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\phi})}\delta_\Lambda\bar{\phi} = \boxed{i(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}) = j^\mu}$$

Remember: j^μ = conserved current

with $\underline{\partial_\mu j^\mu = 0}$ after imposing field equations

$\Rightarrow$ define the conserved charge

because of boundary conditions at ∞

$$0 = \int_{t_1}^{t_2} dx^0 \, \partial_\mu j^\mu = \int_{t_1}^{t_2} dx^0 \Big(\partial_0 \underbrace{\int d^3x \, j^0}_{q(x^0) = q(t)} - \overbrace{\int d^3x \, \partial_i j^i}^{0} \Big)$$

$$0 = q(t_2) - q(t_1) \quad \text{or} \quad q(t_1) = q(t_2)$$

$$\text{or} \quad q(t) = \text{const} \quad (\text{conserved})$$

$$\boxed{q = -\int d^3x \, \mathrm{Im}(\bar{\phi} \, \dot{\phi})}$$

another example of Noether's theorem: Symmetry $\Leftrightarrow$ conserved charges

7.3. From global to local (gauging)

Question: Can we make this global symm. local?

naively $\Lambda \to \Lambda(x^\mu)$ ↯, but then

$$\delta \mathcal{L} = \dots = (\partial_\mu \Lambda) \, j^\mu \neq 0$$

modify (gauge) $\mathcal{L}$ to compensate with additional terms

$$\mathcal{L}_1 = -e \, j^\mu A_\mu$$

new field which we know already from E.D.

remember transformation $A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \Lambda$

$$= A_\mu + \delta A_\mu$$

$$\Rightarrow \boxed{\delta_\Lambda A_\mu = \frac{1}{e} \partial_\mu \Lambda}$$

$$\delta \mathcal{L}_1 = -e \, \delta_\Lambda j^\mu A_\mu - j^\mu \partial_\mu \Lambda$$

term that we want

' but we also get this one :-c with

$$\delta j^\mu = 2\bar\phi\phi\partial^\mu\Lambda \rightsquigarrow \delta(\mathcal{L}+\mathcal{L}_1) = -2eA_\mu\partial^\mu\Lambda\phi\bar\phi$$

Compensate again!

$$\mathcal{L}_2 = e^2 A_\mu A^\mu \phi\bar\phi \Rightarrow \delta\mathcal{L}_2 = 2eA_\mu\partial^\mu\Lambda\phi\bar\phi$$:-)

$$\begin{aligned}\mathcal{L}_{\text{gauged}} &= \mathcal{L}+\mathcal{L}_1+\mathcal{L}_2 \\ &= (\partial_\mu\phi + ieA_\mu\phi)(\partial^\mu\bar\phi - ieA^\mu\bar\phi) - m^2\phi\bar\phi\end{aligned}$$

Symmetries of this Lagrangian are hard to see.
Can we do better?

7.4. Covariant derivatives

$$\boxed{D_\mu\phi := (\partial_\mu + ieA_\mu)\phi}$$

Special because:

$$\begin{aligned}\delta(D_\mu\phi) &= \delta(\partial_\mu\phi) + ie\,\delta A_\mu\phi + ieA_\mu\delta\phi \\ &= -i\partial_\mu(\Lambda\phi) + i\cancel{e}\frac{1}{\cancel{e}}\partial_\mu\Lambda\phi + eA_\mu\Lambda\phi \\ &= -i\Lambda(\partial_\mu\phi + ieA_\mu\phi) = -i\Lambda(D_\mu\phi)\end{aligned}$$

$D_\mu\phi$ transforms like ϕ $(\delta\phi = -i\Lambda\phi)$, namely covariantly

$$\boxed{\mathcal{L}_{\text{gauge}} = D_\mu\phi\,\overline{D_\mu\phi} - m^2\phi\bar\phi}$$

describes complex scalar which is minimally coupled to electromagnetic field

only depends on the charge not higher multipol momenta

7.5. Maxwell theory

But now we have a new field A_μ without kinetic term (no ∂_μ's on A_ν).

Question: Is there a nice (gauge invariant kinetic term) with two derivatives

@ $$[D_\mu, D_\nu]\phi = (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\phi - (\mu \leftrightarrow \nu)$$

$$= \cancel{\partial_\mu \partial_\nu \phi} + ie\,\partial_\mu(A_\nu \phi) + ieA_\mu \partial_\nu \phi - ie^2 A_\mu A_\nu \phi - (\mu \leftrightarrow \nu)$$

$$= ie\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu} \text{ we know from E.D.}}\phi$$

⇝ combing both

$$\boxed{S_{tot} = -1/4\, F_{\mu\nu}F^{\mu\nu} + D_\mu \phi \overline{D^\mu \phi} - m^2 \phi \bar{\phi}}$$

scalar electrodynamics