Strings, Membranes, and a Hidden Symmetry Algebra in Quantum Gravity

Falk Hassler

Based on work with Martin Cederwall, Achilleas Gitsis, Ondřej Hulík, David Osten, Yuho Sakatani and Luca Scala

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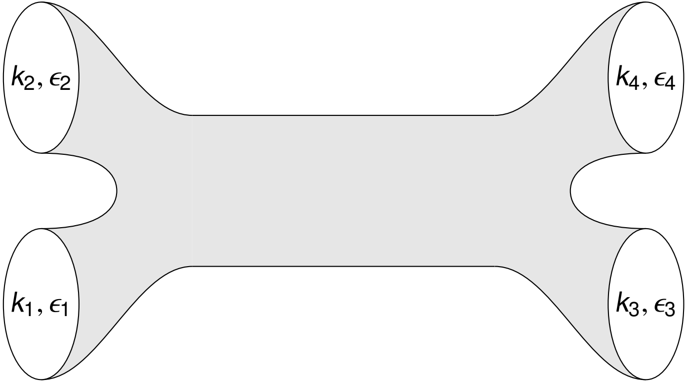
String amplitudes, Effective Field Theory and gravity



massless modes

metric	g_{ij}
gauge potential	B_{ij}
dilaton	ϕ
$\vdots$	

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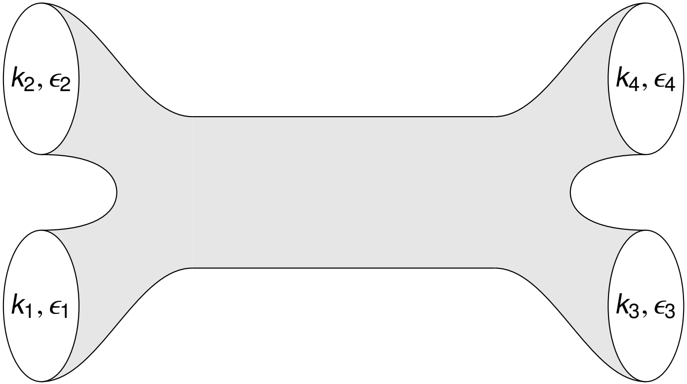

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$$\mathcal{A}^{\text{tree}}(s,t,u) = g_{\text{s}}^2 \frac{(\alpha')^4}{stu} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)} \mathcal{R}^4$$

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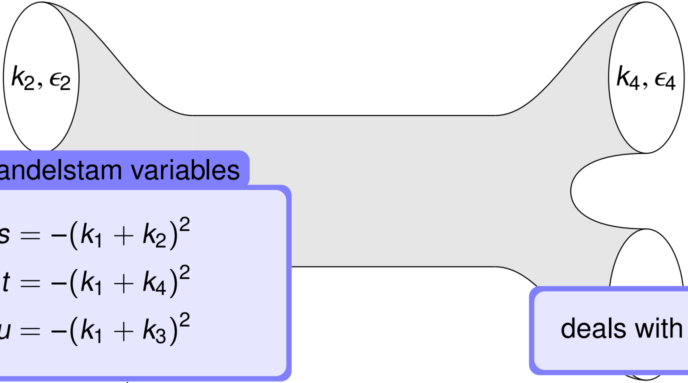

Mandelstam variables

$$s = -(k_1 + k_2)^2$$
$$t = -(k_1 + k_4)^2$$
$$u = -(k_1 + k_3)^2$$

deals with polarizations ϵ_i

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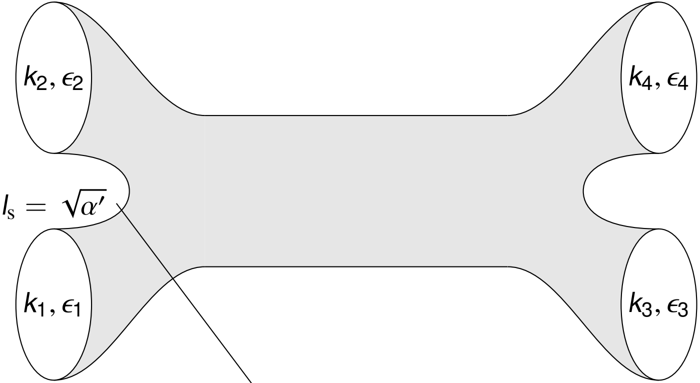

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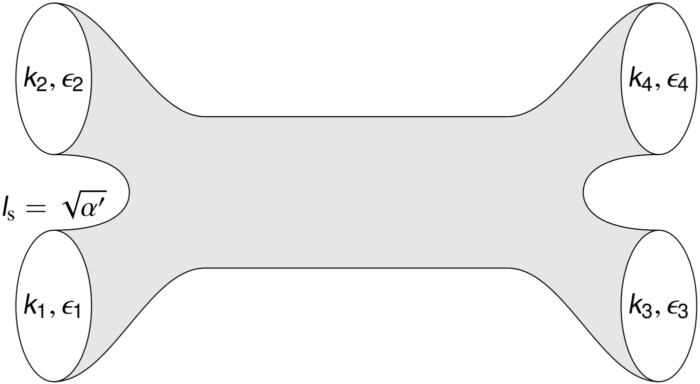

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$$= g_s^2 (\alpha')^2 \left[\frac{1}{stu} + 2\zeta(3)(\alpha')^3 + \zeta(5)(\alpha')^5(s^2+t^2+u^2) + \dots \right] \mathcal{R}^4$$

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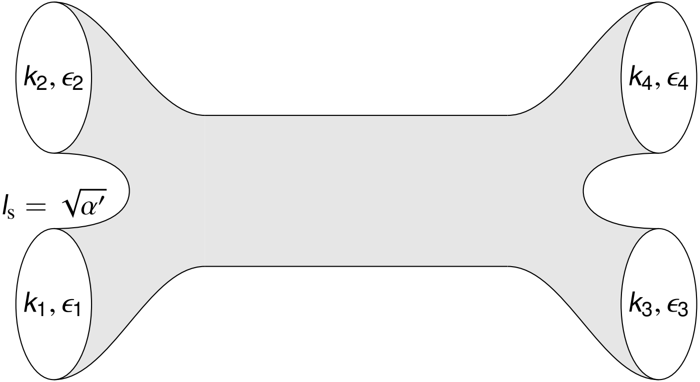

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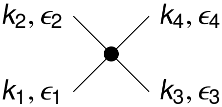

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3. expand action by writing all terms allowed by symmetries (α')

$R_{ijk}{}^{l}, \qquad H_{ijk} = 3\partial_{[i}B_{jk]}, \qquad \phi, \qquad \nabla_i$ building blocks

$$S = \frac{2\pi}{(4\pi^2\alpha')^4}\int \mathrm{d}x^d \sqrt{-g}e^{-2\phi}\left[c_1 R + c_2 \nabla_i\phi\nabla^i\phi + c_3 H_{ijk}H^{ijk} + \alpha'(\dots) + \dots\right]$$

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4. fix their coefficients my matching the amplitudes

$$S = \frac{2\pi}{(4\pi^2\alpha')^4}\int \mathrm{d}x^d \sqrt{-g}e^{-2\phi}\left[R + 4\nabla_i\phi\nabla^i\phi + \tfrac{1}{12}H_{ijk}H^{ijk} + \alpha'(\dots) + \dots\right]$$

Challenges

- explosion of terms

	coeff.	theories
α'^0	3	all
α'^1	8	bos., het.
α'^2	60	bos., het.
α'^3	872	all

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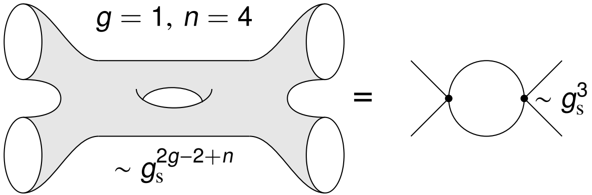

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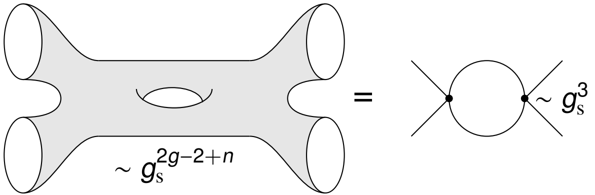

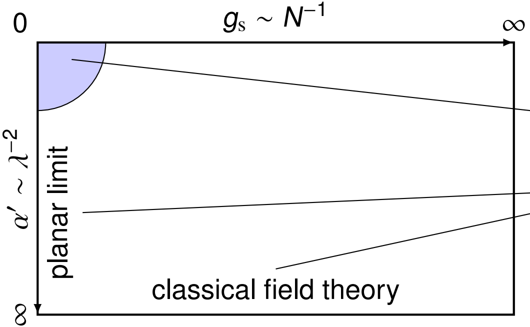

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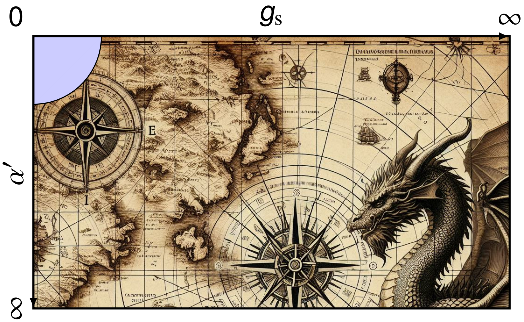

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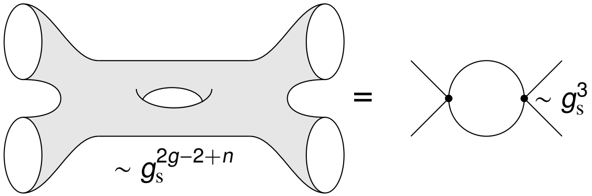

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How to go beyond?

Idea: new symmetries

1. select relevant degrees of freed
2. identify their symmetries*)
3. ...

additional symmetries

- fermionic
- hidden
- spontaneously broken

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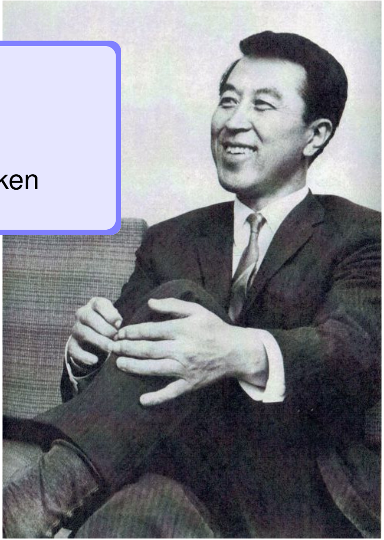

Yoichiro Nambu, 1965

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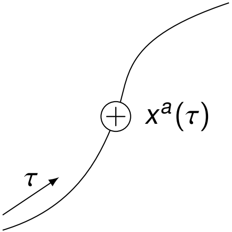
Coset construction (non-linear realization)

- $G =$ the full symmetry group
- $H =$ residual symmetries after spontaneous breaking
- introduce Mauer-Cartan form $\Omega = g^{-1}\mathrm{d}g$ for $g \in G/H$
- expand it in broken generators t_a as $\Omega = \Omega^a t_a + \ldots$

Lagrangian

$$L = L(\Omega^a, D_a)$$

Example: charged particle in an electromagnetic field



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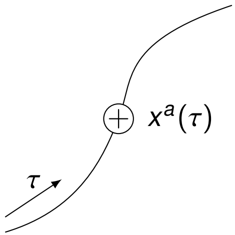

- G is the Poincaré group
- H is the Lorentz group

algebra

P_a	translations
M_{ab}	Lorentz transformations

$$[P_a, P_b] = 0$$

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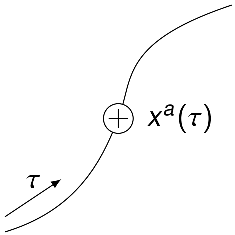

- G is the Poincaré group
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- coset representative $g = e^{x^a P_a}$

$$\Omega = g^{-1}\mathrm{d}g = \Omega^a P_a$$

$$\Omega^a = \dot{x}^a \mathrm{d}\tau$$

algebra

P_a translations
M_{ab} Lorentz transformations

$$[P_a, P_b] = 0$$

- invariant Lagrangian

$$L = m\sqrt{-\Omega^a\Omega_a}$$

- results in field equations

$$\boxed{m\ddot{x}^a = 0}$$

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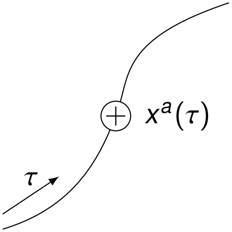

algebra

P_a	translations
M_{ab}	Lorentz transformations
Z_{ab}	constant field strength

$$[P_a, P_b] = Z_{ab}$$

- G is the Maxwell group [Schrader 72]
- H is the Lorentz group
- coset representative $g = e^{x^a P_a} e^{\frac{1}{2}\theta^{ab} Z_{ab}}$

$$\Omega = g^{-1}\mathrm{d}g = \Omega^a P_a + \tfrac{1}{2}\Omega^{ab} Z_{ab}$$

$$\Omega^a = \dot{x}^a \mathrm{d}\tau$$

$$\Omega^{ab} = \left(\dot{\theta}^{ab} + \dot{x}^{[a} x^{b]}\right)\mathrm{d}\tau$$

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$$L = m\sqrt{-\Omega^a \Omega_a} + \frac{1}{2} f_{ab}\Omega^{ab}$$

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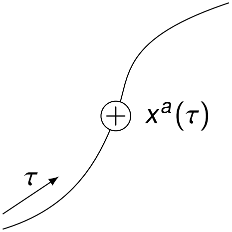

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$$\Omega^a = \dot{x}^a \mathrm{d}$$

$$\Omega^{ab} = \big(\dot{\theta}^{ab}$$

f_{ab} is a Lagrangian multiplier, rendering θ^{ab} non-dynamic.

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Maxwell$_\infty$ algebra, or the road to non-constant fields

1. extend Lie(G) with new generators from P_a-commutator, like

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2. Jacobi identity implies

   $$Y_{[abc]} = [Z_{[ab}, P_{c]}] = [[P_{[a}, P_b], P_{c]}] = 0$$

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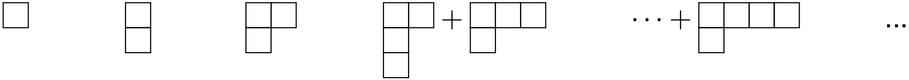
- repeat ℓ-times to get Maxwell$_{\ell+1}$ algebra [Bonanos, Gomis 08; Kleinschmidt, Gomis 17]

P_a $\quad Z_{ab}$ $\quad Y_{abc}$ $\quad S_{abcd}$ $\quad T_{abcde}$ $\quad$ ...



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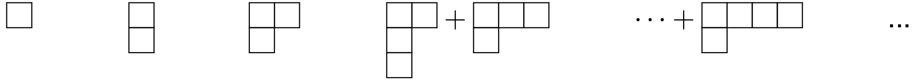
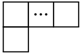


- take subalgebra generated by  to integrate out auxiliary fields $\theta^{\cdots}$

$$m\ddot{x}^a = F^{ab}\dot{x}_b \quad \text{with Taylor expansion} \quad F_{ab} = \sum_{\ell=0}^{\infty} f_{abc_1\ldots c_\ell} x^{c_1} \cdots x^{c_\ell}$$

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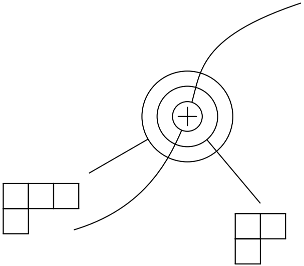

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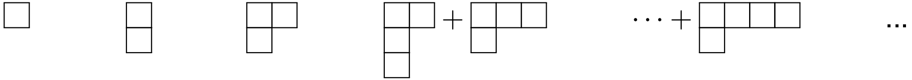
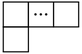


- take subalgebra generated by  to integrate out auxiliary fields $\theta^{\cdots}$

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+

Degrees of freedom in linearized gravity. Part I: the complex

- frame field $e_i^a = \delta_i^a + \varepsilon_i^a$
- symmetries $\delta\varepsilon^a = \mathrm{d}\xi^a + \Lambda^a{}_b \mathrm{d}x^b$

useful relations

$$g_{ij} = \eta_{ab} e_i^a e_j^b$$

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- curvature $$R^a{}_b = \color{red}{\mathrm{d}\omega^a{}_b}$$

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Degrees of freedom in linearized gravity. Part I: the complex

- frame field $$e_i^a = \delta_i^a + \varepsilon_i^a$$
- symmetries $$\delta\varepsilon^a = \mathrm{d}\xi^a + \Lambda^a{}_b \mathrm{d}x^b \qquad \tau(\Lambda^a{}_b)$$
- spin-connection $$\delta\omega^a{}_b = \mathrm{d}\Lambda^a{}_b$$
- torsion $$T^a = \mathrm{d}\varepsilon^a + \omega^a{}_b \wedge \mathrm{d}x^b \qquad \tau(\omega^a{}_b)$$
- curvature $$R^a{}_b = \mathrm{d}\omega^a{}_b$$

$$\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

useful relations

$$g_{ij} = \eta_{ab} e_i^a e_j^b$$

$$\varepsilon^a = \varepsilon_i^a \mathrm{d}x^i$$

$$\nabla_i e_j^a = 0$$

Degrees of freedom in linearized gravity. Part I: the complex

- frame field $\qquad e^a_i = \delta^a_i + \varepsilon^a_i$
- symmetries $\qquad \delta\varepsilon^a = \mathrm{d}\xi^a + \Lambda^a{}_b \mathrm{d}x^b$ $\quad \tau(\Lambda^a{}_b)$
- spin-connection $\qquad \delta\omega^a{}_b = \mathrm{d}\Lambda^a{}_b$
- torsion $\qquad T^a = \mathrm{d}\varepsilon^a + \omega^a{}_b \wedge \mathrm{d}x^b$ $\quad \tau(\omega^a{}_b)$
- curvature $\qquad R^a{}_b = \mathrm{d}\omega^a{}_b$

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

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$$g_{ij} = \eta_{ab} e^a_i e^b_j$$

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Degrees of freedom in linearized gravity. Part I: the complex

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- spin-connection $\qquad \delta\omega^a{}_b = \mathrm{d}\Lambda^a{}_b$
- torsion $\qquad T^a = \mathrm{d}\varepsilon^a + \omega^a{}_b \wedge \mathrm{d}x^b$ $\qquad \tau(\omega^a{}_b)$
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$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

useful relations

$$g_{ij} = \eta_{ab} e^a_i e^b_j$$

$$\varepsilon^a = \varepsilon^a_i \mathrm{d}x^i$$

$$\nabla_i e^a_j = 0$$

$$\mathrm{d}^2 = 0$$

$$\tau^2 = 0$$

$$\mathrm{d} \circ \tau + \tau \circ \mathrm{d} = 0$$

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

- compute cohomology for the exact sequence

$$
0 \longrightarrow \Lambda_{ab} \xrightarrow{\tau} \varepsilon_{ai} \longrightarrow 0
$$

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}$$

- compute cohomology for the exact sequence

$$\varnothing \longrightarrow \overset{\Lambda_{ab}}{\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}} \xrightarrow{\tau} \overset{\varepsilon_{ai}}{\Box \times \Box} \longrightarrow 0$$

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
& C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}$$

- compute cohomology for the exact sequence

$$\emptyset \longrightarrow \underset{\Lambda_{ab}}{\ydiagram{1,1}} \xrightarrow{\tau} \underset{\varepsilon_{ai}}{\ydiagram{1,1} + \ydiagram{2}} \longrightarrow 0$$

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
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Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \overset{\square\!\square}{\varepsilon^a} \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] \arrow[ur, blue, bend right=10] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

- compute cohomology for all diagonal exact sequences

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] \arrow[uurr, blue] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

- compute cohomology for all diagonal exact sequences
- only with Weyl tensor cohomology classes of ε^a and $R^a{}_b$ match

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
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Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\overset{\square}{\xi^a} \arrow[r, "d"] & \overset{\square\square}{\varepsilon^a} \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & \overset{\square\square}{R^a{}_b} \arrow[r] \arrow[ur] & \overset{\square}{\mathrm{BI}(R^a{}_b)} \arrow[r] \arrow[ur] & \cdots \\
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\end{tikzcd}
$$

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- only with Weyl tensor cohomology classes of ε^a and $R^a{}_b$ match

Degrees of freedom in linearized gravity. Part II: cohomology [see i.e. Vasiliev 05]

$$
\begin{tikzcd}
\overset{\square}{\xi^a} \arrow[r, "\mathrm{d}"] & \overset{\square\square}{\varepsilon^a} \arrow[r] & \overset{\substack{0\\ \shortparallel}}{T^a} \arrow[r] \arrow[dl] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & \overset{\square\square}{R^a{}_b} \arrow[r] \arrow[ur] & \overset{\square}{\mathrm{BI}(R^a{}_b)} \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

- compute cohomology for all diagonal exact sequences
- only with Weyl tensor cohomology classes of ε^a and $R^a{}_b$ match
- after imposing torsion constraint ($T^a = 0$), we get

$$
\text{gauge} \longrightarrow \text{dof} \longrightarrow \text{eom} \longrightarrow \text{Bianchi}
$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

$$
\begin{tikzcd}
\xi^a \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab}$$

algebra from point particle

P_a	translations
M_{ab}	Lorentz transformations

$$\begin{tikzcd}
\hat{\xi} & & & & \\
\xi^a \arrow[u] \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[ur, "\tau"] \arrow[r] & \omega^a{}_b \arrow[ur] \arrow[r] & R^a{}_b \arrow[ur] \arrow[r] & \mathrm{BI}(R^a{}_b) \arrow[ur] \arrow[r] & \cdots \\
 & C^a{}_b \arrow[ur] \arrow[r] & \mathrm{BI}(C^a{}_b) \arrow[ur] \arrow[r] & \cdots &
\end{tikzcd}$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

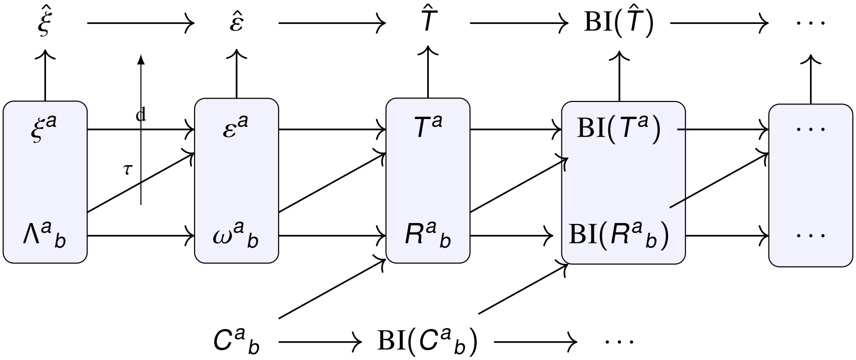
$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab}\,, \qquad \hat{\varepsilon} = \varepsilon^a P_a + \omega^{ab} M_{ab}$$

$$\begin{tikzcd}
\hat{\xi} \arrow[r] & \hat{\varepsilon} & & & \\
\xi^a \arrow[u] \arrow[r, "\mathrm{d}"] & \varepsilon^a \arrow[u] \arrow[r] & T^a \arrow[r] & \mathrm{BI}(T^a) \arrow[r] & \cdots \\
\Lambda^a{}_b \arrow[r] \arrow[ur, "\tau"] & \omega^a{}_b \arrow[r] \arrow[ur] & R^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(R^a{}_b) \arrow[r] \arrow[ur] & \cdots \\
 & C^a{}_b \arrow[r] \arrow[ur] & \mathrm{BI}(C^a{}_b) \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab}\,, \qquad \hat{\varepsilon} = \varepsilon^a P_a + \omega^{ab} M_{ab}\,, \qquad \ldots$$



The background independent version and Cartan geometry

- idea: compress complex into a chain

$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab}\,, \qquad \hat{\varepsilon} = \varepsilon^a P_a + \omega^{ab} M_{ab}\,, \qquad \dots$$

- and use exterior derivative

$$\hat{\mathrm{d}} = \mathrm{d} + \hat{e}\wedge \qquad \text{with}$$

$$\hat{e} = e_i^a P_a \mathrm{d}x^i + \omega^{ab} M_{ab} \qquad \underline{\text{Cartan connection}}$$

$$\hat{\xi} \xrightarrow{\ \hat{\mathrm{d}}\ } \hat{\varepsilon} \longrightarrow \hat{T} \longrightarrow \mathrm{BI}(\hat{T}) \longrightarrow \cdots$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab} \,, \qquad \hat{\varepsilon} = \varepsilon^a P_a + \omega^{ab} M_{ab} \,, \qquad \ldots$$

- and use exterior derivative

$\hat{\mathrm{d}} = \mathrm{d} + \hat{e}\wedge$ with

$\hat{e} = e_i^a P_a \mathrm{d}x^i + \omega^{ab} M_{ab}$ Cartan connection

- substituting $\hat{\varepsilon} \to \hat{e}$ in the chain

$$\hat{\xi} \xrightarrow{\hat{\mathrm{d}}} {\color{red}\hat{e}} \longrightarrow \hat{T} \longrightarrow \mathrm{BI}(\hat{T}) \longrightarrow \cdots$$

The background independent version and Cartan geometry

- idea: compress complex into a chain

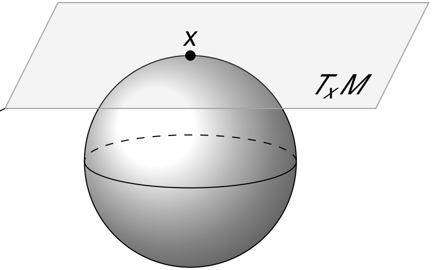
$$\hat{\xi} = \xi^a P_a + \Lambda^{ab} M_{ab}\,, \qquad \hat{\varepsilon} = \varepsilon^a P_a + \omega^{ab} M_{ab}\,, \qquad \ldots$$

- and use exterior derivative

$$\hat{\mathrm{d}} = \mathrm{d} + \hat{e}\wedge \qquad \text{with}$$

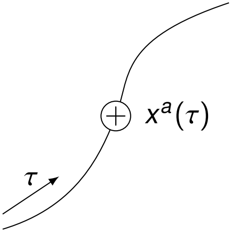
$$\hat{e} = e_i^a P_a \mathrm{d}x^i + \omega^{ab} M_{ab} \qquad \underline{\text{Cartan connection}}$$

- substituting $\hat{\varepsilon} \to \hat{e}$ in the chain
- only input is <u>model space</u> G/H



$$\hat{\xi} \xrightarrow{\hat{\mathrm{d}}} \hat{e} \longrightarrow \hat{T} \longrightarrow \mathrm{BI}(\hat{T}) \longrightarrow \cdots$$

Where is the subalgebra of Maxwell$_\infty$?



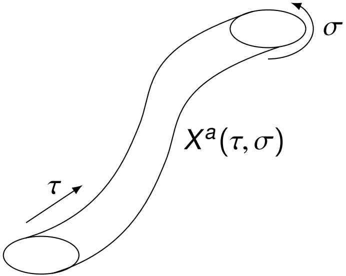
minimal coupling

$$-q\int A\,, \qquad A = A_a \mathrm{d}x^a$$

two-form field strength

$$F = \mathrm{d}A\,, \qquad F = \tfrac{1}{2}F_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b$$

Where is the subalgebra of Maxwell$_\infty$?



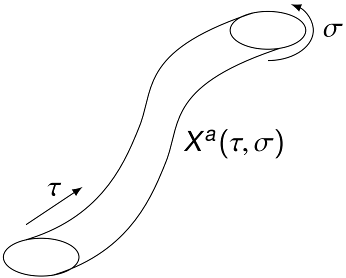
minimal coupling

$$-\int B\,, \qquad B = \tfrac{1}{2} B_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b$$

three-form field strength

$$H = \mathrm{d}B$$

Where is the subalgebra of Maxwell$_\infty$?



minimal coupling

$$-\int B\,, \qquad B = \tfrac{1}{2} B_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b$$

three-form field strength

$$H = \mathrm{d}B$$

diffeomorphism

- gauge transformation

$$\delta B = \mathrm{d}\varphi + L_\xi B$$

- gauge transformation for the gauge transformation

$$\delta\varphi = \mathrm{d}\chi$$

Where is the subalgebra of Maxwell$_\infty$?

$$
\begin{array}{ccc}
\bullet & & \square + \overline{\square} \\
\chi & \xrightarrow{D} & \xi_A
\end{array}
$$

- gauge transformation

$$\delta B = \mathrm{d}\varphi + L_\xi B\,, \qquad \text{with combined parameter} \qquad \xi^A = \begin{pmatrix} \xi^a & \varphi_a \end{pmatrix}$$

- gauge transformation for the gauge transformation

$$\delta\varphi = \mathrm{d}\chi$$

Where is the subalgebra of Maxwell$_\infty$?

$$
\begin{tikzcd}
\bullet & \Box + \overline{\Box} & & & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
 & \Lambda_{AB} \arrow[r] \arrow[ur, "\tau"] & \omega_{ABC} \arrow[r] \arrow[ur] & R_{ABCD} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

- gauge transformation

$$\delta B = \mathrm{d}\varphi + L_\xi B\,, \qquad \text{with combined parameter} \qquad \xi^A = \begin{pmatrix} \xi^a & \varphi_a \end{pmatrix}$$

- gauge transformation for the gauge transformation

$$\delta\varphi = \mathrm{d}\chi$$

Where is the subalgebra of Maxwell$_\infty$?

$$
\begin{tikzcd}
\bullet & \square + \overline{\square} & & & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
& \Lambda_{AB} \arrow[r] \arrow[ur, "\tau"] & \omega_{ABC} \arrow[r] \arrow[ur] & R_{ABCD} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

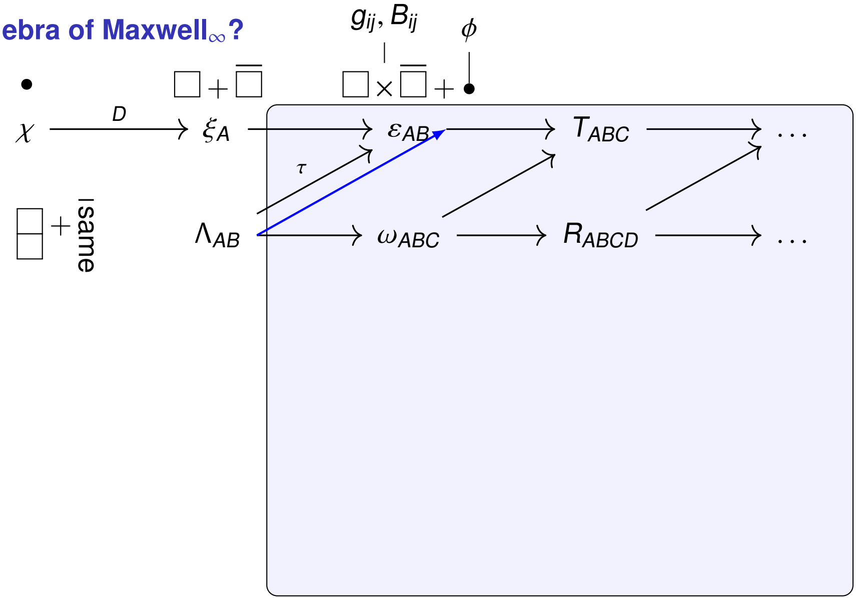
fixed by generalized Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?

$$
\begin{tikzcd}
\bullet & \square + \overline{\square} & \square \times \overline{\square} + \bullet & & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
& \Lambda_{AB} \arrow[r] \arrow[ur, "\tau"] & \omega_{ABC} \arrow[r] \arrow[ur] & R_{ABCD} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

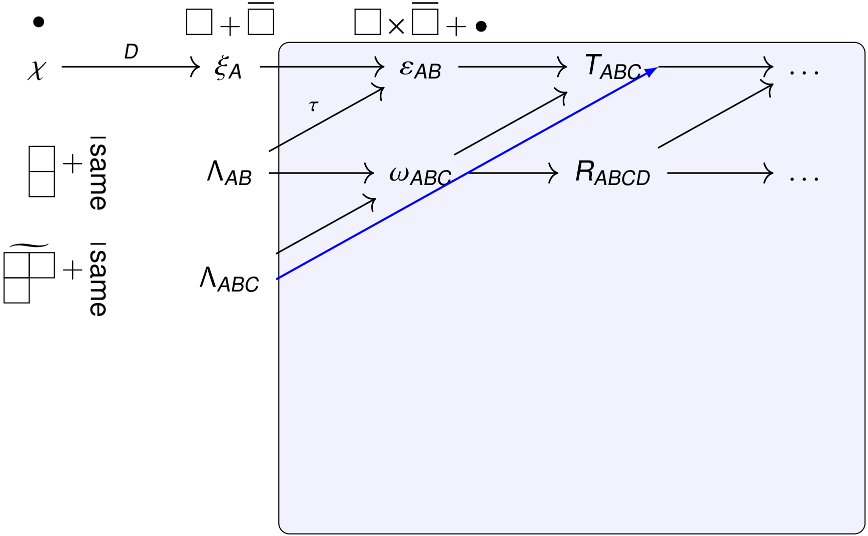
fixed by generalized Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?



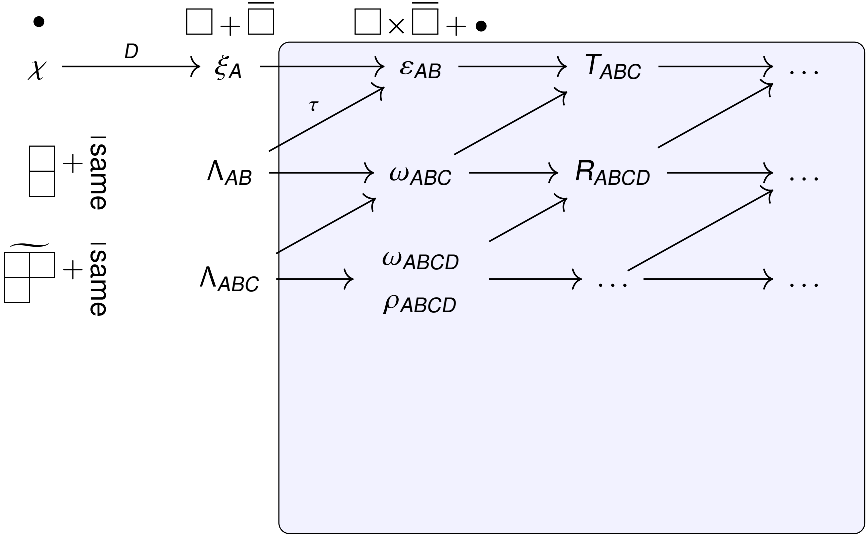
fixed by generalized Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?



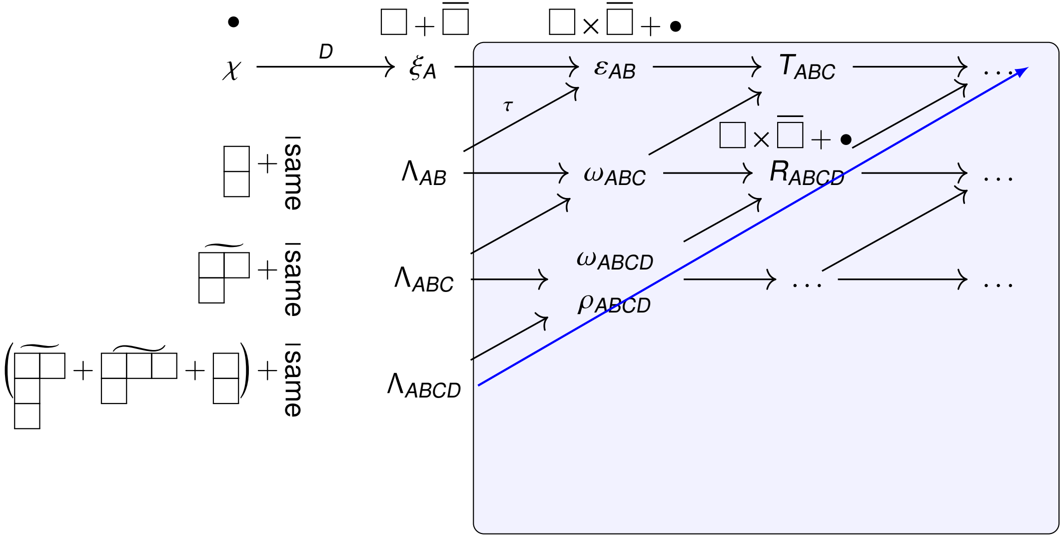
fixed by generalized Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?



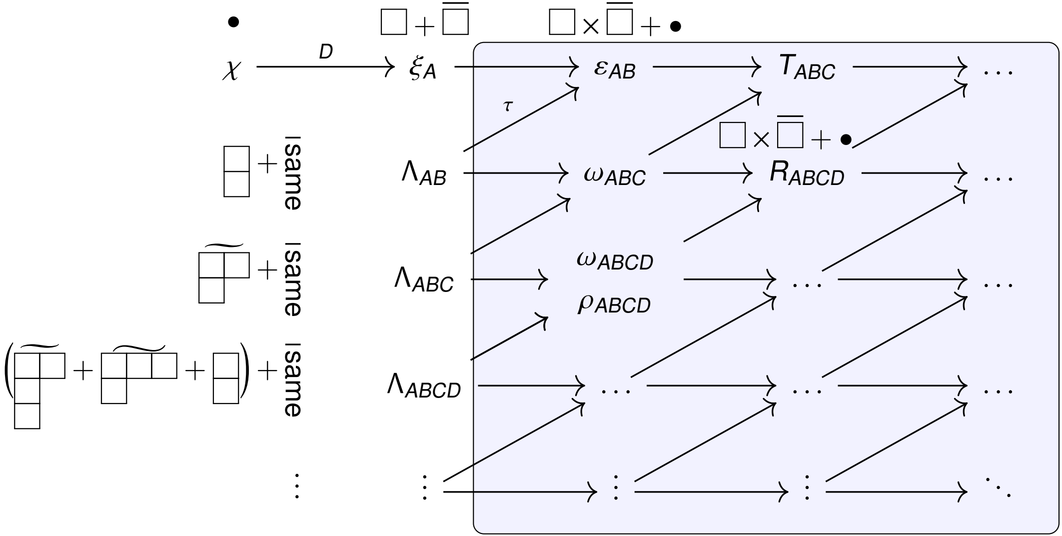
fixed by generalized Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?



fixed by <u>generalized</u> Cartan geometry

Where is the subalgebra of Maxwell$_\infty$?



fixed by generalized Cartan geometry

Relevant features of generalized Cartan geometry [Poláček, Siegel 13; Butter, FH, Pope, Zhang 23; FH, Hulik, Osten 24]

$$\begin{tikzcd}
\hat{\chi} \arrow[rr] & & \hat{\xi} \arrow[dl, dash] \arrow[dr, dash] \arrow[rrr, "\hat{D}"] & & & \hat{E} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[rr] & & \hat{T} \arrow[r] & \mathrm{BI}(\hat{T}) \arrow[r] & \cdots \\
& \xi & & \Lambda & E & \omega & \rho & & &
\end{tikzcd}$$

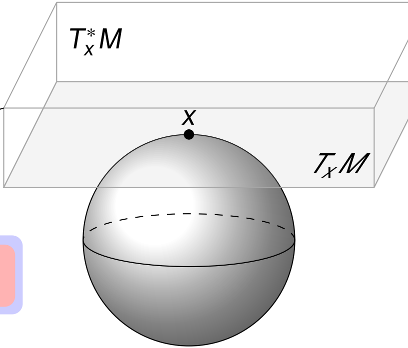
- new connection ρ with corresponding curvature

Relevant features of generalized Cartan geometry [Poláček, Siegel 13; Butter, FH, Pope, Zhang 23; FH, Hulik, Osten 24]

$$
\begin{tikzcd}
\hat{\chi} \arrow[r] & \hat{\xi} \arrow[r, "\hat{D}"] \arrow[dl, dash] \arrow[d, dash] & \hat{E} \arrow[r] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & \hat{T} \arrow[r] & \mathrm{BI}(\hat{T}) \arrow[r] & \cdots \\
\xi & \Lambda & E & \omega & \rho &
\end{tikzcd}
$$

- new connection ρ with corresponding curvature
- model space is double coset $\widetilde{H}\backslash G/H$ generated by

$\dots$	$\tilde{t}^{ABC}$	$\tilde{t}^{AB}$	P_A	t_{AB}	t_{ABC}	$\dots$



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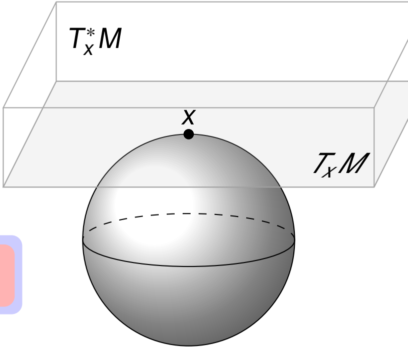
\xi & \Lambda & E \quad \omega & \rho & &
\end{tikzcd}$$



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- model space is double coset $\widetilde{H}\backslash G/H$ generated by

$$\underbrace{\dots \quad \tilde{t}^{ABC} \quad \tilde{t}^{AB} \quad P_A}_{\text{fixed by } \tau^2 = 0 \text{ \& cohomology}} \quad t_{AB} \quad t_{ABC} \quad \dots$$

Relevant features of generalized Cartan geometry [Poláček, Siegel 13; Butter, FH, Pope, Zhang 23; FH, Hulik, Osten 24]

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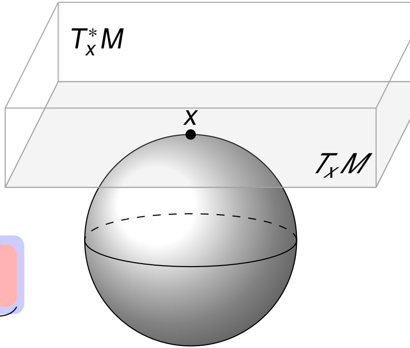
& \xi & & \Lambda & E & \omega & \rho & & &
\end{tikzcd}$$



- new connection ρ with corresponding curvature
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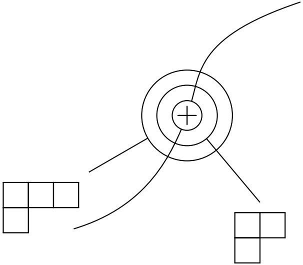
- specified by a symmetric, invariant bilinear form κ on Lie(H)

The point particle on steroids

- we already know the algebra

$$\partial_a \qquad F_{ab} \qquad \partial_c F_{ab} \qquad \partial_d \partial_c F_{ab} \qquad \partial_x \ldots \partial_c F_{ab}$$



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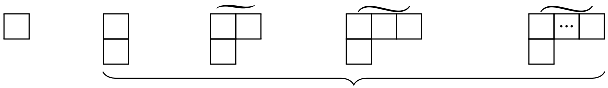
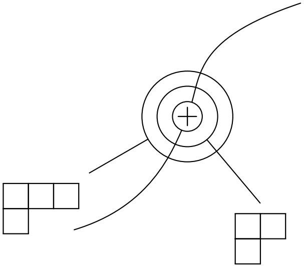


$\tilde{\mathfrak{f}}$ "on-shell algebra"



The point particle on steroids

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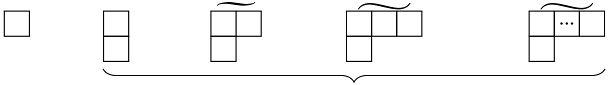
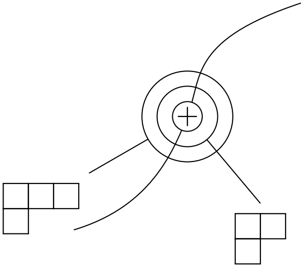
$$\partial_a \qquad F_{ab} \qquad \partial_c F_{ab} \qquad \partial_d \partial_c F_{ab} \qquad \partial_x \dots \partial_c F_{ab}$$



$\widetilde{\mathfrak{f}}$ "on-shell algebra"



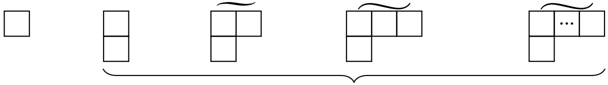
- Lie($\widetilde{H}$) is the free Lie algebra generated by $\widetilde{\mathfrak{f}}$

$$\dots \quad \tilde{t}^{ABC} \quad \tilde{t}^{AB} \quad P_A \quad t_{AB} \quad t_{ABC} \quad \dots$$

The point particle on steroids

- we already know the algebra

$\partial_a \qquad F_{ab} \qquad \partial_c F_{ab} \qquad \partial_d \partial_c F_{ab} \qquad \partial_x \ldots \partial_c F_{ab}$



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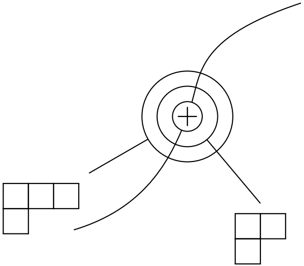

- Lie($\widetilde{H}$) is the free Lie algebra generated by $\tilde{\mathfrak{f}}$

$\ldots \quad \tilde{t}^{ABC} \quad \tilde{t}^{AB} \quad P_A \quad t_{AB} \quad t_{ABC} \quad \ldots$

1. introduce $t_{\hat{A}} = \begin{pmatrix} \tilde{t}^{\alpha} & P_A & t_{\alpha} \end{pmatrix}$ with $[t_{\hat{A}}, t_{\hat{B}}] = f_{\hat{A}\hat{B}}{}^{\hat{C}} t_{\hat{C}}$
2. define $S = \frac{1}{6} f_{\hat{A}\hat{B}\hat{C}} \theta^{\hat{A}} \theta^{\hat{B}} \theta^{\hat{C}}$, and $\{\theta^{\hat{A}}, \theta^{\hat{B}}\} = \eta^{\hat{A}\hat{B}}$
3. solve $\{S, S\} = 0$ (linear order by order)

$$\eta_{\hat{A}\hat{B}} = \begin{pmatrix} 0 & 0 & \delta^{\alpha}_{\beta} \\ 0 & \eta_{AB} & 0 \\ \delta^{\beta}_{\alpha} & 0 & -\kappa_{\alpha\beta} \end{pmatrix}$$

$$\eta^{\hat{A}\hat{B}} = \begin{pmatrix} \kappa_{\alpha\beta} & 0 & \delta^{\beta}_{\alpha} \\ 0 & \eta^{AB} & 0 \\ \delta^{\alpha}_{\beta} & 0 & 0 \end{pmatrix}$$

A tower of corrections

$$
\begin{tikzcd}
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
 & \Lambda_{AB} \arrow[r] \arrow[ur, "\tau"] & \omega_{ABC} \arrow[r] \arrow[ur] & R_{ABCD} \arrow[r] \arrow[ur] & \cdots \\
 & \Lambda_{ABC} \arrow[r] \arrow[ur] & \begin{matrix} \omega_{ABCD} \\ \rho_{ABCD} \end{matrix} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & \Lambda_{ABCD} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots
\end{tikzcd}
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 & \Lambda_{ABC} \arrow[r] \arrow[ur] & \begin{array}{c}\omega_{ABCD}\\ \rho_{ABCD}\end{array} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & \Lambda_{ABCD} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots
\end{tikzcd}
$$

$$\Lambda^{i-2} = \Lambda^{A_1 \dots A_i} t_{A_1 \dots A_i}$$

$$\omega_A^{i-2} = \omega_A^{B_1 \dots B_i} t_{B_1 \dots B_i}$$

$$\vdots$$

$$\rho^{i-2\, j-2} = \tfrac{1}{2} \rho^{A_1 \dots A_i B_1 \dots B_j} t_{A_1 \dots A_i} \wedge t_{B_1 \dots B_j}$$

A tower of corrections

$$\begin{tikzcd}
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
& t_0 \ni \Lambda^0 \arrow[r] \arrow[ur, "\tau"] & \omega^0_A \arrow[r] \arrow[ur] & R^0_{AB} \arrow[r] \arrow[ur] & \cdots \\
& t_1 \ni \Lambda^1 \arrow[r] \arrow[ur] & \begin{matrix} \omega^1_A \\ \rho^{00} \end{matrix} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
& t_2 \ni \Lambda^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
& \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots
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A tower of corrections

$$
\begin{tikzcd}
 & & \square\times\overline{\square}+\bullet & 0 \arrow[d, phantom, "\shortparallel"] & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r, red] & T_{ABC} \arrow[r] \arrow[dl, red, "\square\times\overline{\square}+\bullet"] & \cdots \\
 & t_0 \ni \Lambda^0 \arrow[r] \arrow[ur, "\tau"] & \omega^0_A \arrow[r] \arrow[ur] & R^0_{AB} \arrow[r] \arrow[ur] & \cdots \\
 & t_1 \ni \Lambda^1 \arrow[r] \arrow[ur] & \begin{array}{c}\omega^1_A \\ \rho^{00}\end{array} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & t_2 \ni \Lambda^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
 & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots
\end{tikzcd}
$$

A tower of corrections

$$
\begin{tikzcd}[column sep=large]
 & & \square\times\overline{\square}+\bullet & \overset{0}{\shortparallel} & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r, red] & T_{ABC} \arrow[r] \arrow[dl, red] & \cdots \\
 & t_0 \ni \Lambda^0 \arrow[ur, "\tau"] \arrow[r] & \omega^0_A \arrow[r, red] & R^0_{AB} \arrow[ur, "\square\times\overline{\square}+\bullet"'] \arrow[r] \arrow[dl, red] & \cdots \\
 & t_1 \ni \Lambda^1 \arrow[ur] \arrow[r] & \begin{matrix}\omega^1_A\\ \rho^{00}\end{matrix} \arrow[r] & \cdots \arrow[ur] \arrow[r] & \cdots \\
 & t_2 \ni \Lambda^2 \arrow[ur] \arrow[r] & \cdots \arrow[ur] \arrow[r] & \cdots \arrow[ur] \arrow[r] & \cdots \\
 & \vdots \arrow[ur] \arrow[r] & \vdots \arrow[ur] \arrow[r] & \vdots \arrow[ur] \arrow[r] & \ddots
\end{tikzcd}
$$

A tower of corrections

$$
\begin{tikzcd}[column sep=large]
 & & \square\times\overline{\square}+\bullet & 0 \arrow[d, equal] & \\
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r, red] & T_{ABC} \arrow[r] \arrow[dl, red, "\square\times\overline{\square}+\bullet"] & \cdots \\
 & t_0 \ni \Lambda^0 \arrow[r] \arrow[ur, "\tau"] & \omega^0_A \arrow[r, red] & R^0_{AB} \arrow[r] \arrow[dl, red] \arrow[ur] & \cdots \arrow[ddd, "\sqrt{\alpha'}"] \\
 & t_1 \ni \Lambda^1 \arrow[r] \arrow[ur] & \omega^1_A \ \rho^{00} \arrow[r, red] & \cdots \arrow[r] \arrow[dl, red] \arrow[ur] & \cdots \\
 & t_2 \ni \Lambda^2 \arrow[r] \arrow[ur] & \cdots \arrow[r, red] & \cdots \arrow[r] \arrow[dl, red] \arrow[ur] & \cdots \\
 & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r, red] & \vdots \arrow[r] \arrow[ur] & \ddots
\end{tikzcd}
$$

A tower of corrections

$$R^0_{AB} = \mathrm{d}\omega^0_A + \underbrace{\sum_{i,j}[\omega^i_A, \omega^j_B]|_{t_0} + \dots}_{\text{non-linear terms from gen. Cartan}}$$

$$
\begin{tikzcd}
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots & \\
& t_0 \ni \Lambda^0 \arrow[r] \arrow[ur, "\tau"] & \omega^0_A \arrow[r] \arrow[ur] & R^0_{AB} \arrow[r] \arrow[ur] & \cdots \arrow[ddd, "\sqrt{\alpha'}"] & \\
& t_1 \ni \Lambda^1 \arrow[r] \arrow[ur] & \begin{matrix}\omega^1_A \\ \rho^{00}\end{matrix} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots & \\
& t_2 \ni \Lambda^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots & \\
& \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots &
\end{tikzcd}
$$

A tower of corrections

$$
\begin{tikzcd}
\chi \arrow[r, "D"] & \xi_A \arrow[r] & \varepsilon_{AB} \arrow[r] & T_{ABC} \arrow[r] & \cdots \\
& t_0 \ni \Lambda^0 \arrow[r] \arrow[ur, "\tau"] & \omega^0_A \arrow[r, blue] \arrow[ur] & R^0_{AB} \arrow[r] \arrow[ur] & \cdots \arrow[ddd, "\sqrt{\alpha'}"] \\
& t_1 \ni \Lambda^1 \arrow[r] \arrow[ur] & \begin{matrix}\omega^1_A \\ \rho^{00}\end{matrix} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
& t_2 \ni \Lambda^2 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \cdots \\
& \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \vdots \arrow[r] \arrow[ur] & \ddots
\end{tikzcd}
$$

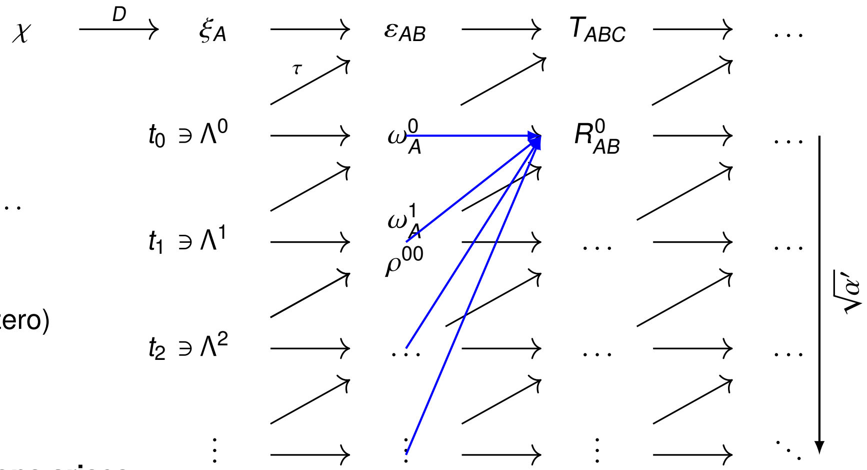
$$R^0_{AB} = \mathrm{d}\omega^0_A + \sum_{i,j} [\omega^i_A, \omega^j_B]|_{t_0} + \ldots$$

H is graded (κ vanishes)

$[t_i, t_j] \subset t_{i+j}$

no α'-corrections

A tower of corrections



$$R^0_{AB} = \mathrm{d}\omega^0_A + \sum_{i,j} [\omega^i_A, \omega^j_B]|_{t_0} + \dots$$

H is a filtration (κ non-zero)

$$[t_i, t_j] \subset \bigcup_{k \le i+j} t_k$$

a tower of α'-corrections arises

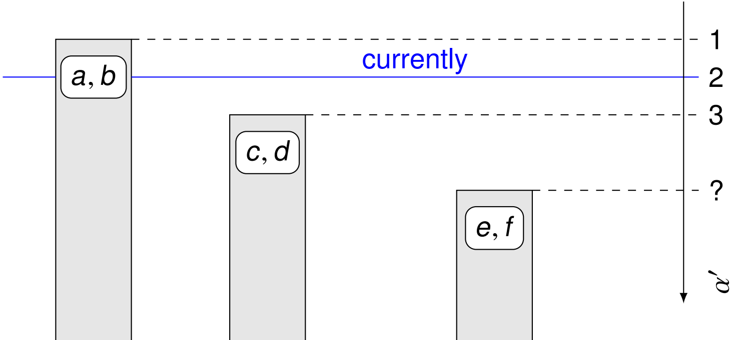
A skyline and an evaded no-go theorem

- admissible κ's are parameterized by $\kappa = \kappa(\underbrace{a, b}_{\text{have to be there}}, c, d, e, f, \dots)$

[Achilleas Gitsis, FH 24]

A skyline and an evaded no-go theorem

- admissible κ's are parameterized by $\kappa = \kappa(\underline{a, b}, c, d, e, f, \dots)$
- each parameter creates a tower of α'-corrections

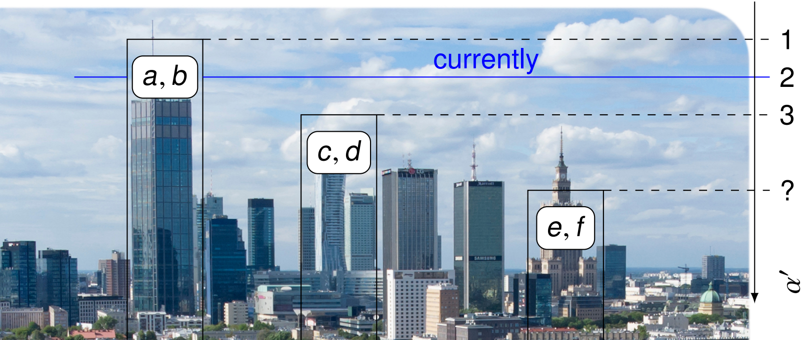


	theories
α'^0	all
α'^1	bos., het.
α'^2	bos., het.
α'^3	all

A skyline and an evaded no-go theorem

- admissible κ's are parameterized by $\kappa = \kappa(\underline{a, b}, c, d, e, f, \dots)$
- each parameter creates a tower of α'-corrections
- no-go for α'^3-tower from deformed O(d)×O(d) symmetry [Hsia, Kamal, Wulff 24]

No problem, we don't need this symmetry!



	theories
α'^0	all
α'^1	bos., het.
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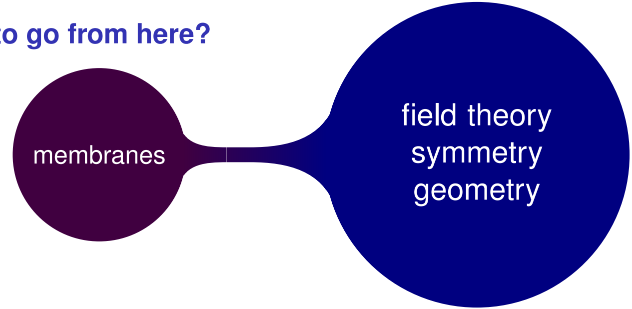
α'^3	all

Where to go from here?

field theory
symmetry
geometry

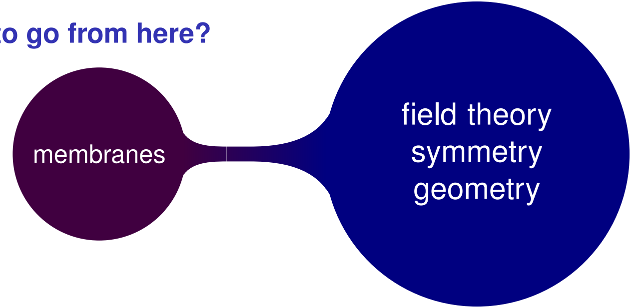
Where to go from here?



generalized Cartan geometry:

$$\begin{tikzcd}
\hat{\chi} \arrow[rr] & & \hat{\xi} \arrow[rrr, "\hat{D}"] \arrow[dl, dash] \arrow[dr, dash] & & & \hat{E} \arrow[rr] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \hat{T} \arrow[r] & \cdots \\
& \xi & & \Lambda & E & \omega & \rho & &
\end{tikzcd}$$

Where to go from here?

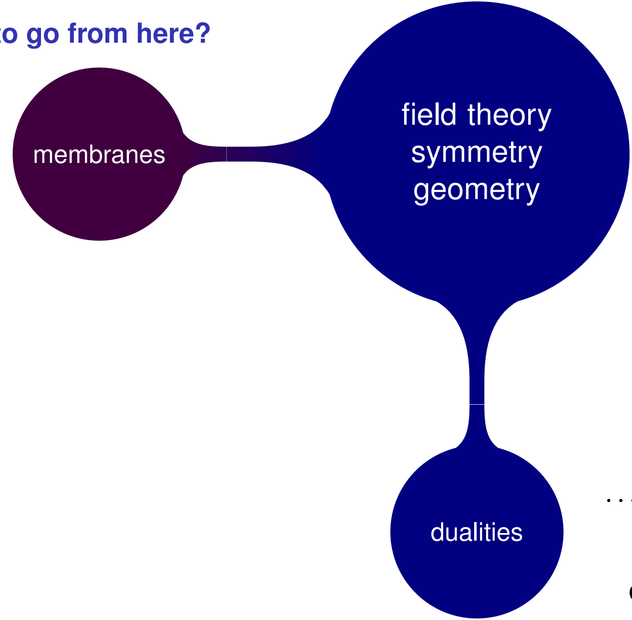


exceptional Cartan geometry: [FH, Yuho Sakatani 23]

tensor hierarchy

$$\begin{tikzcd}
\cdots \arrow[r] & \hat\chi \arrow[r] & \hat\xi \arrow[r, "\hat D"] \arrow[dl, dash] \arrow[d, dash] & \hat E \arrow[r] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] \arrow[drrr, dash] & \hat T \arrow[r] & \cdots & \\
& \xi & \Lambda & E & \omega & \rho & \cdots
\end{tikzcd}$$

Where to go from here?

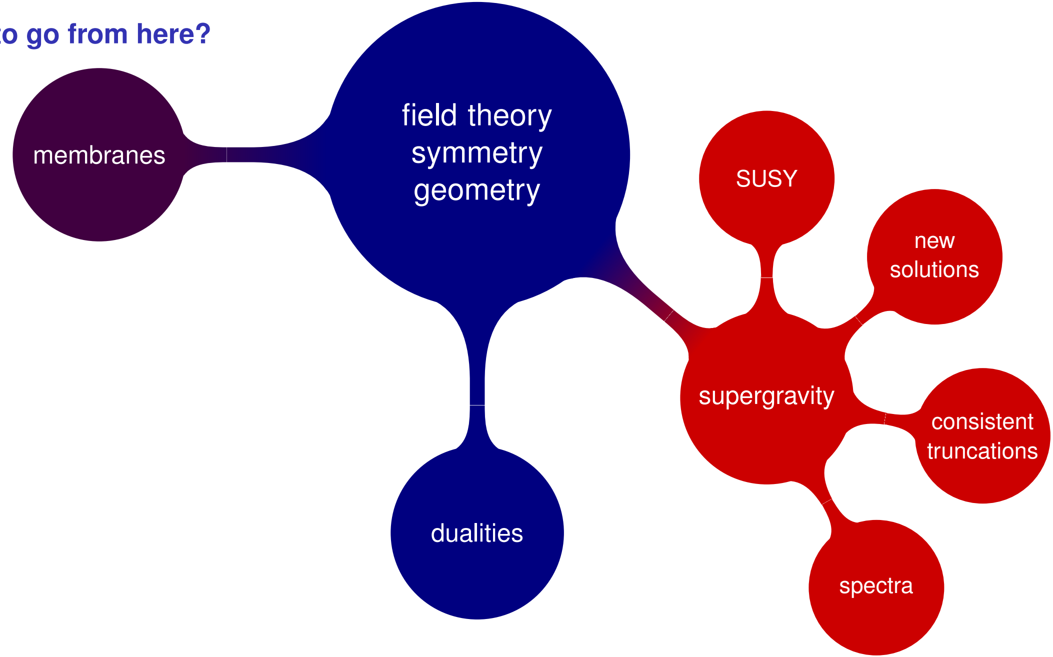


constant

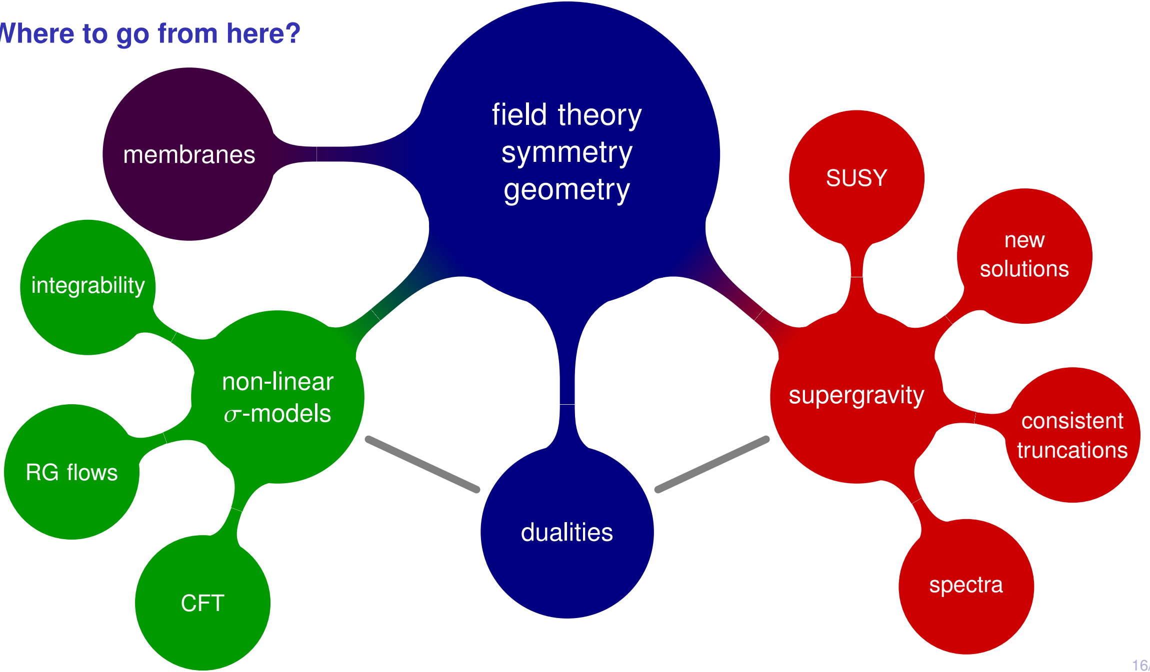
$$\cdots \longrightarrow \hat{E} \xrightarrow{\hat{D}} \hat{T} \longrightarrow \cdots$$

different realizations [FH 17; ...]

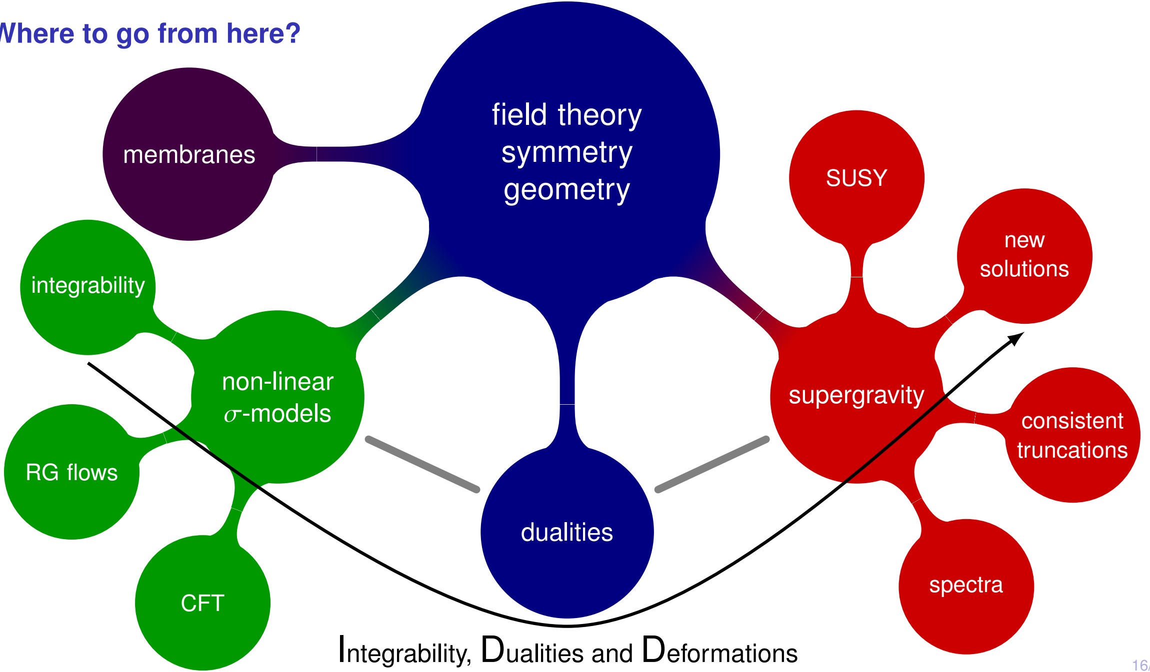
Where to go from here?



Where to go from here?



Where to go from here?



Integrability, Dualities and Deformations

Where to go from here

membranes

integrability

non-… σ-m…

RG flows

CFT

…Y

new solutions

…avity

consistent truncations

spectra

Integrability, Dualities and Deformations 2025

@ NORDITA (Stockholm, Sweden)

- 08.09-12.09.2025 workshop
- 15.09-19.09.2025 conference

https://indico.fysik.su.se/event/8807

Where to go from here?