Integrability, Poisson-Lie Symmetry and Double Field Theory

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and

1502.02428 with Pascal du Bosque, Dieter Lüst and Ralph Blumenhagen

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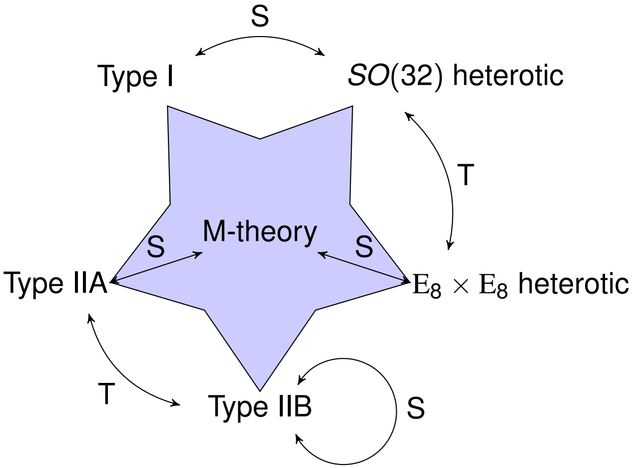
Canonical motivation for Exception/Double Field Theory

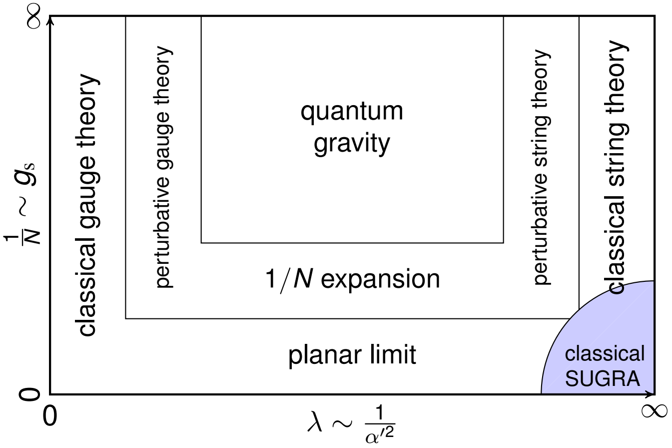


T = T-duality

S = S-duality

But there is also another interesting story. . .

AdS/CFT correspondence



Outline

1. **Motivation**
2. **Integrability and AdS/CFT**
3. **Poisson-Lie Symmetry**
4. **Double Field Theory on Drinfeld doubles**
5. **Summary**

Integrability

or how to “solve” 4D maximal SYM completely

Anomalous dimension in 4D $\mathcal{N} = 4$ SYM

- CFT two point function of primaries

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta}}$$

- scaling dimension gets renormalized

$$\Delta = \Delta_0 + \lambda\Delta_1 + \dots$$

- example single trace operator $\operatorname{Tr} Z^L \quad Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$S = \int d^4x \operatorname{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi_i D^\mu\phi^i - \frac{g^2}{4}[\phi_i,\phi_j][\phi^i,\phi^j] + \text{fermions}\right)$$

- $\Delta_0 = L$ what about $\Delta_1, \dots$

- more general single trace operators with $(L-M)\times Z$ and $M \times W = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$

SU(2) spin chain and the Bethe ansatz

- $\Delta_1 \leftrightarrow$ eigenvalues of the Heisenberg spin chain

  $$H = 2\sum_{l=1}^{L}\left(\frac{1}{4} - \vec{S}_l\vec{S}_{l+1}\right) \qquad S_l = \frac{1}{2}\vec{\sigma}_l$$

  $Z = \uparrow$, $W = \downarrow$, and for $L = 3$ $\mathrm{Tr}\, ZZW = |\uparrow\uparrow\downarrow\rangle$

- Bethe ansatz gives rise to eigenvalues and vectors
- just possible because spin chain is **integrable**
- integrability is so powerful that it also to find all corrections

  $\Delta_1, \Delta_2, \Delta_3 \ldots$

Where is the integrability in string theory?

Ingredients for classical/quantum integrability:

1. Hamiltonian/Hamilton operator
2. Poisson-bracket/commutator
3. Lax pair

- example Principal Chiral Model (PCM)

$$S = \frac{1}{2}\int d^2\sigma \, \mathrm{Tr}(g^{-1}\partial_+ g \, g^{-1}\partial_- g)$$

$$H = \frac{1}{2}\int d\sigma \, \mathrm{Tr}(j_0^2 + j_1^2) \quad j_0 = g^{-1}\partial_\tau g \quad j_1 = g^{-1}\partial_\sigma g$$

$$\{j_{0\,a}(\sigma), j_{0\,b}(\sigma')\} = f_{ab}{}^c j_{0\,c} \qquad L_\pm(\lambda) = \frac{j_0 \pm j_1}{1 \pm \lambda}$$

$$\{j_{0\,a}(\sigma), j_{1\,b}(\sigma')\} = f_{ab}{}^c j_{1\,c} + \delta_{ab}$$

$$\{j_{1\,a}(\sigma), j_{1\,b}(\sigma')\} = 0$$

Let's generalize this construction!

- Hamiltonian (Poisson-Lie σ-model) :

$$H = \frac{1}{2}\int d\sigma j_A(\sigma)\mathcal{H}^{AB} j_B(\sigma)$$

- Poisson-bracket:

$$\{j_A(\sigma), j_B(\sigma')\} = F_{AB}{}^C j_C(\sigma)\delta(\sigma-\sigma') + \eta_{AB}\delta'(\sigma-\sigma')$$

- Lax pair:

$$L_\pm(\lambda) = \frac{\mathcal{J} \pm \mathcal{R}}{1 \pm \lambda}$$

All know integrable 2D non-linear σ-models can be brought in this form. They are fixed completely by specifying the **constants** $\mathcal{H}^{AB}$ and $\mathcal{F}_{AB}{}^C$.

Examples:

- η-deformation
  - with/without WZW term
  - on group mainfolds
  - and coset spaces
- λ-deformation

Poisson-Lie symmetry

Poisson as in Poisson-bracket:
required for the Hamiton formalism

Lie as in Lie-algbra:
e.g. required for Lax's equation

$$\frac{dL}{dt} = [P, L]$$

Drinfeld double [Drinfeld, 1988]

Definition: A **Drinfeld double** is a $2D$-dimensional Lie group $\mathcal{D}$, whose Lie-algebra $\mathfrak{d}$

1. has an ad-invariant bilinear for $\langle\cdot\,,\,\cdot\rangle$ with signature (D, D)
2. admits the decomposition into two maximal isotropic subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$

- $\begin{pmatrix} t^a & t_a \end{pmatrix} = t_A \in \mathfrak{d}\,, \quad t_a \in \mathfrak{g}$ and $t^a \in \tilde{\mathfrak{g}}$
- $\langle t_A, t_B \rangle = \eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta_a^b & 0 \end{pmatrix}$
- $[t_A, t_B] = F_{AB}{}^C t_C$ with non-vanishing commutators

$$[t_a, t_b] = f_{ab}{}^c t_c \qquad [t_a, t^b] = \tilde{f}^{bc}{}_a t_c - f_{ac}{}^b t^c$$

$$[t^a, t^b] = \tilde{f}^{ab}{}_c t^c$$

- ad-invariance of $\langle\cdot\,,\,\cdot\rangle$ implies $F_{ABC} = F_{[ABC]}$

Poisson-Lie Symmetry [Klimcik and Severa, 1995]

- 2D σ-model on target space M with action

  $S(E, M) = \int dz d\bar{z}\, E_{ij} \partial x^i \bar{\partial} x^j$
- $E_{ij} = g_{ij} + B_{ij}$ captures metric and two-from field on M
- inverse of E_{ij} is denoted as E^{ij}
- *left* invariant vector field ${v_a}^i$ on G is the inverse transposed of *right* invariant Maurer-Cartan form $t_a {v^a}_i dx^i = dg\, g^{-1}$
- adjoint action of $g \in G$ on $t_A \in \mathfrak{d}$: $\mathrm{Ad}_g\, t_A = g t_A g^{-1} = {M_A}^B t_B$
- analog for $\tilde{G}$

Definition: $S(E, \mathcal{D}/\tilde{G})$ has **Poisson-Lie Symmetry** if

$$E^{ij} = {v_c}^i {M_a}^c (M^{ae} {M^b}_e + E_0^{ab}) {M_b}^d {v_d}^j$$

holds, where E_0^{ab} is constant and invertible with the inverse $E_{0\,ab}$.

Immediate consequence: Poisson-Lie T-duality

- exchanging G and $\tilde{G}$ results in dual σ-model with

$$\tilde{E}^{ij} = \tilde{v}^{ci}\tilde{M}^a{}_c(\tilde{M}_{ae}\tilde{M}_b{}^e + E_{0\,ab})\tilde{M}^b{}_d\tilde{v}^{dj}$$

- captures
  - abelian T-d. — G abelian and $\tilde{G}$ abelian
  - non-abelian T-d. — G non-abelian and $\tilde{G}$ abelian

  [Ossa and Quevedo, 1993;Giveon and Rocek, 1994; Alvarez, Alvarez-Gaume, and Lozano, 1994;...]
- dual σ-models related by canonical transformation
  [Klimcik and Severa, 1995;Klimcik and Severa, 1996;Sfetsos, 1998]

$\rightarrow$ equivalent at the classical level

- preserves conformal invariance at one-loop
  [Alekseev, Klimcik, and Tseytlin, 1996;Sfetsos, 1998;...;Jurco and Vysoky, 2017]
- dilaton transformation [Jurco and Vysoky, 2017]

$$\phi = -\frac{1}{2}\log\left|\det\left(1 + \tilde{g}_0^{-1}(\tilde{B}_0 + \Pi)\right)\right|$$
$$\tilde{\phi} = -\frac{1}{2}\log\left|\det\left(1 + g_0^{-1}(B_0 + \tilde{\Pi})\right)\right|$$

SUGRA

- DFT makes PL-Symmetry manifest
- consistent tructions are central
- get the dialton, R/R sector nearly for free

Additional structure on the Drinfeld double

[Blumenhagen, Hassler, and Lüst, 2015, Blumenhagen, Bosque, Hassler, and Lüst, 2015]

- *right* invariant vector $E_A{}^I$ field on $\mathcal{D}$ is the inverse transposed of *left* invariant Maurer-Cartan form $t_A E^A{}_I dX^I = g^{-1} dg$
- two η-compatible, covariant derivatives[1]
  1. flat derivative

     $$D_A V^B = E_A{}^I \partial_I V^B - w F_A V^B, \qquad F_A = D_A \log \left|\det(E^B{}_I)\right|$$

  2. convenient derivative

     $$\nabla_A V^B = D_A V^B + \tfrac{1}{3} F_{AC}{}^B V^C$$

- generalized metric $\mathcal{H}_{AB}$ ($w = 0$)

$$\mathcal{H}_{AB} = \mathcal{H}_{(AB)}, \qquad \mathcal{H}_{AC} \eta^{CD} H_{DB} = \eta_{AB}$$

- generalized dilaton d with e^{-2d} scalar density of weight $w = 1$
- triple $(\mathcal{D}, \mathcal{H}_{AB}, d)$ captures the doubled space of DFT

[1] definitions here just for quantities with flat indices

Double Field Theory for $(\mathcal{D}, \mathcal{H}_{AB}, d)$ [Blumenhagen, Bosque, Hassler, and Lüst, 2015]

see also [Vaisman, 2012; Hull and Reid-Edwards, 2009;Geissbuhler, Marques, Nunez, and Penas, 2013; Cederwall, 2014; . . .]

- action ($\nabla_A d = -\frac{1}{2} e^{2d} \nabla_A e^{-2d}$)

$$S_{\mathrm{NS}} = \int_{\mathcal{D}} d^{2D} X e^{-2d} \Big(\frac{1}{8} \mathcal{H}^{CD} \nabla_C \mathcal{H}_{AB} \nabla_D \mathcal{H}^{AB} - \frac{1}{2} \mathcal{H}^{AB} \nabla_B \mathcal{H}^{CD} \nabla_D \mathcal{H}_{AC} - 2 \nabla_A d \nabla_B \mathcal{H}^{AB} + 4 \mathcal{H}^{AB} \nabla_A d \nabla_B d + \frac{1}{6} F_{ACD} F_B{}^{CD} \mathcal{H}^{AB} \Big)$$

- generalized diffeomorphisms

$$\mathcal{L}_\xi V^A = \xi^B \nabla_B V^A + \left(\nabla^A \xi_B - \nabla_B \xi^A \right) V^B + w \nabla_B \xi^B V^A$$

- $2D$-diffeomorphisms

$$L_\xi V^A = \xi^B D_B V^A + w D_B \xi^B V^A$$

- global O(D,D) transformations

$$V^A \to T^A{}_B V^B \quad \text{with} \quad T^A{}_C T^B{}_D \eta^{CD} = \eta^{AB}$$

- section condition (SC)

$$\eta^{AB} D_A \cdot \; D_B \cdot = 0$$

Symmetries of the action

- S_{NS} invariant for $X^I \to X^I + \xi^A E_A{}^I$ and
  1. $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + \mathcal{L}_\xi \mathcal{H}^{AB}$ and $e^{-2d} \to e^{-2d} + \mathcal{L}_\xi e^{-2d}$
  2. $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + L_\xi \mathcal{H}^{AB}$ and $e^{-2d} \to e^{-2d} + L_\xi e^{-2d}$

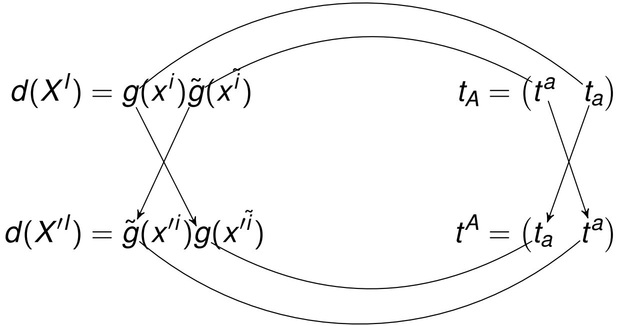
object	gen.-diffeomorphisms	$2D$-diffeomorphisms	global O(D,D)
$\mathcal{H}_{AB}$	tensor	scalar	tensor
$\nabla_A d$	not covariant	scalar	1-form
e^{-2d}	scalar density (w=1)	scalar density (w=1)	invariant
η_{AB}	invariant	invariant	invariant
$F_{AB}{}^C$	invariant	invariant	tensor
$E_A{}^I$	invariant	vector	1-form
S_{NS}	invariant	invariant	invariant
SC	invariant	invariant	invariant
D_A	not covariant	covariant	covariant
∇_A	not covariant	covariant	covariant

manifest (last two columns)

Poisson-Lie T-duality: 1. Solve SC [Hassler, 2016]

- fix D physical coordinates x^i from $X^I = \begin{pmatrix} x^i & x^{\tilde{i}} \end{pmatrix}$ on $\mathcal{D}$

  such that $\eta^{IJ} = E_A{}^I \eta^{AB} E_B{}^J = \begin{pmatrix} 0 & \cdots \\ \cdots & \cdots \end{pmatrix} \rightarrow$ SC is solved
- fields and gauge parameter depend just on x^i
- only *two* SC solutions, relate them by symmetries of DFT



Poisson-Lie T-duality: 2. As manifest symmetry of DFT

- same structure as in the original paper [Klimcik and Severa, 1995]
- duality target spaces arise as different solutions of the SC

Poisson-Lie T-duality:

- 2D-diffeomorphisms $X^I \to X'^I(X^1, \dots X^{2D})$ with $d(X^I) = d(X'^I)$
- global O(D,D) transformation $t_A \to \eta^{AB} t_B$

manifest symmtries of DFT

- for abelian T-duality $X^I \to X'^I = X^I$

$\to$ no 2D-diffeomorphisms needed, only global O(D,D) transformation

Poisson-Lie Symmetry is a manifest symmetry of DFT

Equivalence to supergravity: 1. Generalized parallelizable spaces

[Lee, Strickland-Constable, and Waldram, 2014]

- generalized tangent space element $V^{\hat{I}} = \begin{pmatrix} V^i & V_i \end{pmatrix}$
- generalized Lie derivative

$$\widehat{\mathcal{L}}_\xi V^{\hat{I}} = \xi^{\hat{J}} \partial_{\hat{J}} V^{\hat{I}} + (\partial^{\hat{I}} \xi_{\hat{J}} - \partial_{\hat{J}} \xi^{\hat{I}}) V^{\hat{J}} \qquad \text{with} \qquad \partial_{\hat{I}} = \begin{pmatrix} 0 & \partial_i \end{pmatrix}$$

Definition: A manifold M which admits a globally defined generalized frame field $\widehat{E}_A{}^{\hat{I}}(x^i)$ satisfying

1. $\widehat{\mathcal{L}}_{\widehat{E}_A} \widehat{E}_B{}^{\hat{I}} = F_{AB}{}^C \widehat{E}_C{}^{\hat{I}}$

   where $F_{AB}{}^C$ are the structure constants of a Lie algebra $\mathfrak{h}$
2. $\widehat{E}_A{}^{\hat{I}} \eta^{AB} \widehat{E}_B{}^{\hat{J}} = \eta^{\hat{I}\hat{J}} = \begin{pmatrix} 0 & \delta_i^j \\ \delta^i_j & 0 \end{pmatrix}$

is a **generalized parallelizable space** $(M, \mathfrak{h}, \widehat{E}_A{}^{\hat{I}})$.

Equivalence to supergravity: 2. Generalized metric and dilaton

[Klimcik and Severa, 1995; Hull and Reid-Edwards, 2009; du Bosque, Hassler, Lüst, 2017]

- Drinfeld double $\mathcal{D} \to$ two generalized parallelizable spaces:

$$(D/\tilde{G}, \mathfrak{d}, \widehat{E}_A{}^{\widehat{I}}) \quad \text{and} \quad (D/G, \mathfrak{d}, \widetilde{\widehat{E}}_A{}^{\widehat{I}})$$

$$\widehat{E}_A{}^{\widehat{I}} = M_A{}^B \begin{pmatrix} v^b{}_i & 0 \\ 0 & v_b{}^i \end{pmatrix}_B{}^{\widehat{I}} \qquad \widetilde{\widehat{E}}_A{}^{\widehat{I}} = \tilde{M}_{AB} \begin{pmatrix} \tilde{v}_{bi} & 0 \\ 0 & \tilde{v}^{bi} \end{pmatrix}^{B\widehat{I}}$$

- express $\mathcal{H}^{AB}$ in terms of the generalized $\widehat{\mathcal{H}}^{\hat{I}\hat{J}}$ on $TD/\tilde{G} \oplus T^*D/\tilde{G}$

$$\mathcal{H}^{AB} = \widehat{E}^A{}_{\hat{I}} \widehat{\mathcal{H}}^{\hat{I}\hat{J}} \widehat{E}^B{}_{\hat{J}} \qquad \text{with} \qquad \widehat{\mathcal{H}}^{\hat{I}\hat{J}} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lk} & -B_{ik} g^{kl} \\ g^{ik} B_{kj} & g^{ij} \end{pmatrix}$$

- express d in terms of the standard generalized dilaton $\widehat{d}$

$$d = \widehat{d} - \frac{1}{2} \log |\det \tilde{v}_{ai}|$$

$$\widehat{d} = \phi - 1/4 \log \left|\det g_{ij}\right|$$

- plug into the DFT action S_{NS}

Equivalence to supergravity: 3. IIA/B bosonic sector action

- if G and $\tilde{G}$ are unimodular

$$S_{\mathrm{NS}} = V_{\tilde{G}} \int d^D x \, e^{-2\widehat{d}} \Big(\frac{1}{8} \widehat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{K}} \widehat{\mathcal{H}}_{\hat{I}\hat{J}} \partial_{\hat{L}} \widehat{\mathcal{H}}^{\hat{I}\hat{J}} - 2 \partial_{\hat{I}} \widehat{d} \partial_{\hat{J}} \widehat{\mathcal{H}}^{\hat{I}\hat{J}} - \frac{1}{2} \widehat{\mathcal{H}}^{\hat{I}\hat{J}} \partial_{\hat{J}} \widehat{\mathcal{H}}^{\hat{K}\hat{L}} \partial_{\hat{L}} \widehat{\mathcal{H}}_{\hat{I}\hat{K}} + 4 \widehat{\mathcal{H}}^{\hat{I}\hat{J}} \partial_{\hat{I}} \widehat{d} \partial_{\hat{J}} \widehat{d} \Big)$$

- $V_{\tilde{G}} = \int_{\tilde{G}} d\tilde{x}^D \det \tilde{v}_{ai}$ volume of group $\tilde{G}$.
- equivalent to IIA/B NS/NS sector action
  [Hohm, Hull, and Zwiebach, 2010; Hohm, Hull, and Zwiebach, 2010]

$$S_{\mathrm{NS}} = V_{\tilde{G}} \int \mathrm{d}^D x \sqrt{\det(g_{ij})} e^{-2\phi} \big(\mathcal{R} + 4 \partial_i \phi \partial^i \phi - \frac{1}{12} H_{ijk} H^{ijk} \big)$$

- holds for all $\mathcal{H}_{AB}(x^i)$ / $\widehat{\mathcal{H}}^{\hat{I}\hat{J}}$
- only D-diffeomorphisms and B-field gauge trans. as symmetries

- similar story for R/R sector

Restrictions on $\mathcal{H}_{AB}$ and d to admit Poisson-Lie Symmetry

- in general $\mathcal{H}_{AB}(x^i) \xrightarrow{\text{Poisson-Lie T-duality } (2D\text{-diff.})} \mathcal{H}_{AB}(x'^i, x'^{\tilde{i}})$
- $x'^{\tilde{i}}$ part not compatible with ansatz for SC solutions $\rightarrow$ avoid it

A doubled space $(\mathcal{D}, \mathcal{H}_{AB}, d)$ admits Poisson-Lie T-dual supergravity descriptions iff

1. $L_\xi \mathcal{H}_{AB} = 0 \quad \forall \xi \quad \rightarrow \quad D_A \mathcal{H}_{AB} = 0$
2. $L_\xi d = 0 \quad \forall \xi \quad \rightarrow \quad D_A e^{-2d} = 0$

Application: Dilaton profile

- $D_A e^{-2d} = 0 \quad \rightarrow \quad \partial_I(\underbrace{2d + \log|\det v| + \log|\det \tilde{v}|}_{=2\phi_0 = \text{const.}}) = 0$
- $d = \phi - 1/4 \log|\det g| - \frac{1}{2}\log|\det \tilde{v}| \quad \rightarrow \quad \phi = \phi_0 + \frac{1}{4}\log|\det g| - \frac{1}{2}\log|\det v|$
- $g = v^T e^T e v \quad \text{with} \quad \begin{cases} (\tilde{B}_0 + \tilde{g}_0)^{ab} = E^{0\,ab} \\ \Pi^{ab} = M^{ac} M^b{}_c \\ e^{-1}e^{-T} = \tilde{g}_0 - (\tilde{B}_0 + \Pi)\tilde{g}_0^{-1}(\tilde{B}_0 + \Pi) \\ \tilde{e}_0^T \tilde{e}_0 = \tilde{g}_0 \\ e^{-T} = \tilde{e}_0 + \tilde{e}_0^{-T}(\tilde{B}_0 + \Pi) \end{cases}$
- $\phi = \phi_0 + \frac{1}{2}\log|\det e| = \phi_0 - \frac{1}{2}\log|\det \tilde{e}_0| - \frac{1}{2}\log\left|\det\left(1 + \tilde{g}_0^{-1}(\tilde{B}_0 + \Pi)\right)\right|$
- reproduces [Jurco and Vysoky, 2017]

Summary

- DFT, Poisson-Lie T-duality and Drinfeld doubles fit together naturally
- interpretation of doubled space does not require winding modes anymore (phase space perspective instead)
- various new directions for research in DFT
  - connection to integrability in SUGRA
  - Drinfeld doubles $\rightarrow$ quantum groups $\rightarrow$ rich mathematical structure
  - new way to organized α' corrections?
  - implication for consistent truncation
  - branes in curved space [Klimcik, and Severa, 1996 (D-branes)]?
- facilitates new applications
  - integrable deformations of 2D σ-models
  - solution generating technique
  - explore underlying structure of AdS/CFT

Summary

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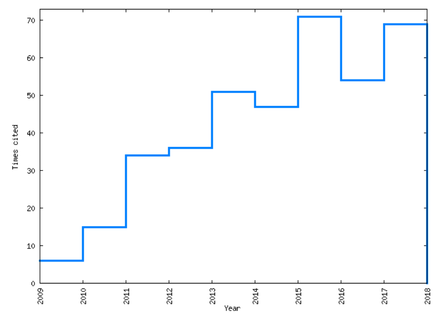
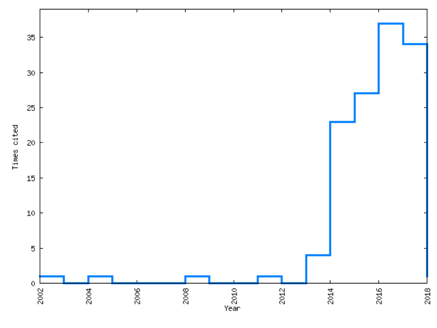
- interpretation of doubled space does not require winding modes



Hull and Zwiebach, 2009



Klimcik, 2002

  - solution generating technique
  - explore underlying structure of AdS/CFT