

# Feynman diagrams and path integral expansion

considers  $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4!} \phi^4$ .

We want to compute:

$$\begin{aligned} \langle \phi_1 \phi_2 \rangle &= \langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle \\ &= \frac{\int \mathcal{D}\phi e^{iS} \phi_1 \phi_2}{\int \mathcal{D}\phi e^{iS}} = \frac{I_2}{I_0} \end{aligned}$$

$$\begin{aligned} I_0 &= \int \mathcal{D}\phi e^{iS_0} \left( 1 - \frac{i\lambda}{4!} \int d^4y \phi^4(y) + \mathcal{O}(\lambda^2) \right) \\ &= Z_0[0] \left( 1 - \frac{i\lambda}{4!} \int d^4y \langle \phi^4(y) \rangle_0 + \mathcal{O}(\lambda^2) \right) \end{aligned}$$

with  $Z_0[0] = \int \mathcal{D}\phi e^{iS_0}$  and  $\langle \dots \rangle_0 = \frac{\int \mathcal{D}\phi e^{iS_0} \dots}{Z_0[0]}$

$$I_2 = \dots = Z_0[0] \left( \underbrace{\langle \phi_1 \phi_2 \rangle_0}_A - \frac{i\lambda}{4!} \int d^4y \underbrace{\langle \phi_1 \phi_2 \phi^4(y) \rangle_0}_{-B} \right)$$

Same as for  $I_0$ .

$$\langle \phi_1 \phi_2 \rangle = \frac{I_2}{I_1} = \frac{A + B\lambda + \mathcal{O}(\lambda^2)}{1 + C\lambda + \mathcal{O}(\lambda^2)} = A + (B - AC)\lambda + \mathcal{O}(\lambda^2)$$

$$\begin{aligned} &= \langle \phi_1 \phi_2 \rangle_0 - \frac{i\lambda}{4!} \int d^4y \langle \phi_1 \phi_2 \phi^4(y) \rangle_0 \\ &\quad + \langle \phi_1 \phi_2 \rangle_0 \frac{i\lambda}{4!} \int d^4y \langle \phi^4(y) \rangle_0 + \mathcal{O}(\lambda^2) \end{aligned}$$

$\langle \phi_1 \phi_2 \rangle_0 = D_F(x_1 - x_2)$  Feynman propagator of free theory

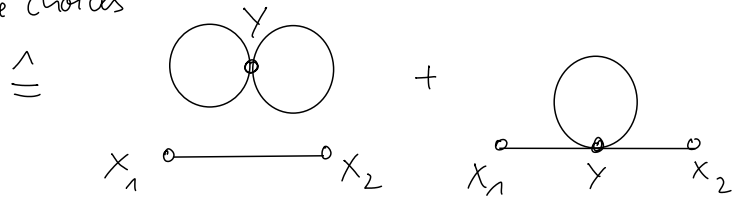
$$\begin{aligned} \langle \phi^4(y) \rangle_0 &= \langle \underbrace{\phi \phi}_{\text{---}} \underbrace{\phi \phi}_{\text{---}} \rangle + \langle \underbrace{\phi \phi \phi \phi}_{\text{---}} \rangle + \langle \underbrace{\phi \phi \phi \phi}_{\text{---}} \rangle \\ &= 3 D_F(y - y)^2 \text{ by Wick's theorem} \end{aligned}$$

$$\langle \phi_1 \phi_2 \phi^4(y) \rangle_0 = \dots = 3 D_F(x_1 - x_2) D_F^2(y - y) +$$

Wick's theorem  $12 D_F(x_1 - y) D_F(x_2 - y) D_F(y - y)$

4 possible choices  
 $\phi_1 \phi_2 \phi \dots \phi$   
 3 possible choices

$$\nearrow = 4 \cdot 3$$



$$\langle \phi_1 \phi_2 \rangle = D_F(x_1 - x_2) + \frac{1}{2} \int d^4 y D_F(x_1 - y) D_F(x_2 - y) D_F(y - y)$$

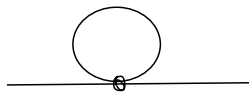
unconnected diagram drops out

We can recover this result from Feynman rules using the symmetry factor

$$S = \left( \frac{1}{4!} \right)^V \frac{C}{V!}$$

$V =$  number of vertices  
 $C =$  number of contractions

i.e. for

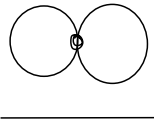


$$V = 1$$

$$C = 4 \cdot 3 = 12$$

$$S = \frac{12}{4!} = \frac{1}{2} //$$

or



$$V = 1$$

$$C = 3$$

$$S = \frac{3}{4!} = \frac{1}{6} //$$

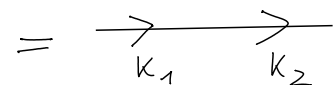
In momentum space

$$\int d^4 x_1 d^4 x_2 e^{i(k_1 x_1 - k_2 x_2)} D_F(x_1 - x_2)$$

$$= \int d^4 x_1 d^4 x_2 \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + i\epsilon} e^{i[x_1(k_1 - p) + x_2(p - k_2)]}$$

$$= (2\pi)^4 \int d^4 p \delta(k_1 - p) \delta(p - k_2) \frac{1}{p^2 + i\epsilon}$$

$$= (2\pi)^4 \delta(k_1 - k_2) \frac{1}{k_1^2 + i\epsilon} //$$



## Momentum conservation

$$\begin{aligned} & \int d^4x_1 d^4x_2 d^4y D_F(x_1-y) D_F(x_2-y) D_F(y-y) e^{i(x_1 k_1 - x_2 k_2)} \\ &= \int d^4x_1 d^4x_2 d^4y \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{4 \cdot 3}} \frac{1}{p_1^2 + i\epsilon} \frac{1}{p_2^2 + i\epsilon} \frac{1}{p_3^2 + i\epsilon} \cdot \\ & \quad \exp \left[ x_1 \cdot (k_1 - p_1) - x_2 \cdot (k_2 + p_2) + y \cdot (p_1 + p_2) \right] \\ &= \int d^4p_1 d^4p_2 d^4p_3 \delta(k_1 - p_1) \delta(k_2 + p_2) \delta(p_1 + p_2) \cdot \\ & \quad \left[ (p_1^2 + i\epsilon)(p_2^2 + i\epsilon)(p_3^2 + i\epsilon) \right]^{-1} \\ & \quad \hookrightarrow k_1 = p_1 \quad k_2 = -p_2 \\ &= \delta(k_1 - k_2) \int d^4p \frac{1}{(k_1^2 + i\epsilon)^2 (k_2^2 + i\epsilon) (p^2 + i\epsilon)} \end{aligned}$$

Conclusion:

Knowing the Feynman rules in momentum space from the lecture, we can immediately write down the correlator  $\langle \phi_1 \phi_2 \rangle$  up to order  $\mathcal{O}(\lambda^2)$ .