Bicrossproduct structure of ϱ-Poincaré and the associated $\star$-product

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Summary

1. The ϱ-Poincaré quantum groups
2. Structure and properties of the quantum groups
3. Conclusions : comparing the models

The ϱ-Poincaré quantum groups

The ϱ-Minkowski Spacetime

The 4D ϱ-Minkowski spacetime, $\mathcal{M}_\varrho$, is defined by the commutation relations

$$[\hat{x}^1, \hat{x}^0] = i\varrho\hat{x}^2 \,,$$
$$[\hat{x}^2, \hat{x}^0] = -i\varrho\hat{x}^1 \,,$$

all other commutators being zero.

These are not invariant under the classical Poincaré transformations

$$\hat{x}`^\mu = \Lambda^\mu{}_\nu \hat{x}^\nu + a^\mu .$$

To solve this problem : deform the Poincaré group to a quantum group, P_ϱ.

The new Poincaré map will be

$$\hat{x}`^\mu = \hat{\Lambda}^\mu{}_\nu \otimes \hat{x}^\nu + \hat{a}^\mu \otimes 1,$$

$$\hat{\Lambda}^\mu{}_\nu, \hat{a}^\mu \in P_\varrho, \qquad \hat{x}^\nu \in \mathcal{M}_\varrho .$$

A short review of quantum groups

To build P_ϱ we have to quantize the parameters $\Lambda^\mu{}_\nu$, a^μ.

1. We consider $C(P)$,
2. we take the classical r-matrix correspondent to the ϱ-deformation :[1]

$$r_\varrho = i\varrho(P_0 \wedge M_{12}) \ ,$$

3. we define a Poisson-Sklyanin bracket via r,
4. we quantize via canonical quantization

$$\Lambda^\mu{}_\nu, a^\mu \to \hat{\Lambda}^\mu{}_\nu, \hat{a}^\mu$$
$$\{,\} \to -\frac{i}{\hbar}[,] \ .$$

Equipped with trivial cosector and antipode, this structure is an Hopf algebra called the quantum group $C_\varrho(P)$.

1. J. Lukierski, A. Nowicki, H. Ruegg, *New quantum poincaré algebra and κ-deformed field theory*, Physics Letters B 293 (1992) 344.

The quantum group $C_\varrho(P)$

The quantum group is, then, given by [2, 3]

$$
\begin{aligned}
[a^\mu, a^\nu] &= i\varrho[\delta^\nu{}_0(a^2\delta^\mu{}_1 - a^1\delta^\mu{}_2) - \delta^\mu{}_0(a^2\delta^\nu{}_1 - a^1\delta^\nu{}_2)]\,,\\
[\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma] &= 0\,,\\
[\Lambda^\mu{}_\nu, a^\rho] &= i\varrho\,[\Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu}) - \delta^\rho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2)]\,\,,\\
\Delta(\Lambda^\mu{}_\nu) &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu\,,\\
\Delta(a^\mu) &= \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes 1\,,\\
\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu\,,\\
\varepsilon(a^\mu) &= 0\,,\\
S(\Lambda^\mu{}_\nu) &= (\Lambda^{-1})^\mu{}_\nu\,,\\
S(a^\mu) &= -a^\nu(\Lambda^{-1})^\mu{}_\nu\,.
\end{aligned}
$$

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2. F. Lizzi, P. Vitale, *Time Discretization From Noncommutativity*, Phys. Lett. B818 (2021), 136372.
3. F. Lizzi, LS, P. Vitale, *Localization and observers in ϱ-Minkowski spacetime*, Phys. Rev. D 106 (2022) 025023.

The quantum algebra $U_\varrho(\mathfrak{p})$

Algebraically-dual to $\mathcal{C}_\varrho(P)$ there is another quantum group, usually called a *quantum algebra* : $U_\varrho(\mathfrak{p})$.

Since r_ϱ satisfies a CYBE there is an easy way to obtain it (*non-standard quantization*) :

1 we consider $U(\mathfrak{p})$,

2 we promote it to a trivial Hopf algebra,

3 we deform this Hopf algebra via a Drinfel'd twist map [4]

$$\mathcal{F}_\varrho = e^{\frac{i\varrho}{2}[P_0 \wedge M_{12}]} ,$$

applying it to the coproducts and antipodes, leaving all the other structure maps unchanged.

The full structure is way too long for these slides !

4. M. D. Ćirić, N. Konjik, A. Samsarov, *Noncommutative scalar quasinormal modes of the Reissner–Nordström black hole*, Class. Quant. Grav. 35 (2018) 175005.

Issues and caveats

When talking about quantum groups there are some subtleties :

1. in general there is not a unique way to obtain these structures, and the procedure depends on the form of the spacetime we are considering,
2. the quantum group associated to a noncommutative spacetime is unique up to isomorphisms ; this means that we can have a lot of different structures leaving the commutation relations covariant.

For our purposes, the point [1] is not relevant, since in the ϱ-case there are two standard ways to obtain the quantum group and the quantum algebra without any problems.

Point [2] is more subtle, and we will discuss it later during the presentation.

Structure and properties of the quantum groups

The bicrossproduct construction

An interesting kind of quantum groups are called bicrossproduct quantum groups.[5]

Given two Hopf algebras $\mathcal{X}, \mathcal{A}$, a bicrossproduct Hopf algebra, $\mathcal{X} \bowtie \mathcal{A}$, is the tensor product $\mathcal{X} \otimes \mathcal{A}$ endowed with two structure maps :

$$\triangleleft : \mathcal{A} \otimes \mathcal{X} \to \mathcal{A},$$
$$\beta : \mathcal{X} \to \mathcal{A} \otimes \mathcal{X},$$

such that

$$(a \cdot b) \triangleleft x = (a \triangleleft x_{(1)})(b \triangleleft x_{(2)}),$$
$$(id \otimes \beta) \circ \beta = (\Delta \otimes id) \circ \beta.$$

It is the analog of a semidirect product.

5. S. Majid, H. Ruegg, *Bicrossproduct structure of kappa Poincaré group and noncommutative geometry*, Phys. Lett. B 334 (1994) 348.

Actions and coactions

The action and coaction are relevant to build the global Hopf structure :

$$\begin{aligned}
\mu((x \otimes a), (y \otimes b)) &= (x \otimes a) \cdot (y \otimes b) = xy_{(1)} \otimes (a \triangleleft y_{(2)})b, \\
1_{\mathcal{X} \triangleright\!\!\blacktriangleleft \mathcal{A}} &= 1_{\mathcal{X}} \otimes 1_{\mathcal{A}}, \\
\Delta(x \otimes a) &= (x_{(1)} \otimes {x_{(2)}}^{(\bar{1})} a_{(1)}) \otimes ({x_{(2)}}^{(\bar{2})} \otimes a_{(2)}), \\
\varepsilon(x \otimes a) &= \varepsilon(x)\varepsilon(a), \\
S(x \otimes a) &= (1_{\mathcal{X}} \otimes S(x^{(\bar{1})} a)) \cdot (S(x^{(\bar{2})}) \otimes 1_{\mathcal{A}}),
\end{aligned}$$

where $x, y \in \mathcal{X}$, $a, b \in \mathcal{A}$ and $\Delta(h) = \sum_i {h_{(1)}}_i \otimes {h_{(2)}}_i = h_{(1)} \otimes h_{(2)}$ and $\beta(x) = x^{(\bar{1})} \otimes x^{(\bar{2})}$.

From the first one $\rightarrow$ the right action gives the commutators ;

from the third and the fifth $\rightarrow$ the left coaction gives the coproducts and antipodes.

Structure and properties of $C_\varrho(P)$

$C_\varrho(P)$ is indeed of the bicrossproduct type. [6]

We start with the undeformed $C(SO(1,3))$ and a deformed $\mathcal{T}^*_\varrho$ given by

$$
\begin{aligned}
[x^\mu, x^\nu] &= i\varrho[\delta^\nu{}_0(x^2\delta^\mu{}_1 - x^1\delta^\mu{}_2) - \delta^\mu{}_0(x^2\delta^\nu{}_1 - x^1\delta^\nu{}_2)] \ , \\
\Delta(x^\mu) &= x^\mu \otimes 1 + 1 \otimes x^\mu \ , \\
S(x^\mu) &= -x^\mu \ , \\
\varepsilon(x^\mu) &= 0 \ .
\end{aligned}
$$

We consider the action and coaction maps

$$
\begin{aligned}
1 \otimes (\Lambda^\mu{}_\nu \triangleleft x^\rho) &= -[x^\rho \otimes 1, 1 \otimes \Lambda^\mu{}_\nu], \\
\beta(x^\mu) &= \Lambda^\mu{}_\nu \otimes x^\nu .
\end{aligned}
$$

6. G. Fabiano, G. Gubitosi, F. Lizzi, LS, P. Vitale, *Bicrossproduct vs. twist quantum symmetries in noncommutative geometries : the case of ϱ-Minkowski*, J. High Energ. Phys. 2023, 220 (2023).

Structure and properties of $\mathcal{C}_\varrho(P)$

Equipped with these structures, we can check that all the axioms are satisfied and define the bicrossproduct quantum group

$$\mathcal{C}_\varrho(P) = \mathcal{T}_\varrho^* \bowtie \mathcal{C}(SO(1,3)).$$

This structure allows for the definition of the noncommutative spacetime quotienting the quantum group.

Classical commutative case : $\mathcal{T}^* = P/SO(1,3) \cong \mathcal{M}$.

Noncommutative case : $\mathcal{T}_\varrho^* \cong \mathcal{M}_\varrho$ as can be seen modding out the quantum group by the Lorentz subalgebra : the commutator of $\mathcal{T}_\varrho^*$ is, in fact, the covariant form of the ϱ-Minkowski commutation relations.

For $U_\varrho(\mathfrak{p})$ the discussion is more complex.

Structure and properties of $U_\varrho(\mathfrak{p})$

$U_\varrho(\mathfrak{p})$, although dual to $\mathcal{C}_\varrho(P)$, does not satisfy the bicrossprodcut properties.

For instance

$$\Delta_{\mathcal{F}} M_{30} = M_{30} \otimes 1 + 1 \otimes M_{30} + \frac{\varrho}{2} P_3 \otimes M_{12} - \frac{\varrho}{2} M_{12} \otimes P_3 \ ,$$

is not compatible with the bicross coproducts linked with a coaction map, due to P_3 in the right side of $\otimes$.

But from the Hopf algebra theory : a quantum algebra of the bicrossproduct type dual to $\mathcal{C}_\varrho(P)$ should exist.

There are a lot of quantum algebras related one to the other via nonlinear change of bases, all isomorphic one to the other.

We want to find a nonlinear change of basis that brings the quantum algebra in the bicrossproduct form.

Structure and properties of $U_\varrho(\mathfrak{p})$

Inspired by the similar case of κ-poincaré we perform the following change of generators :

$$
\begin{aligned}
\widetilde{P}_0 &= P_0,\\
\widetilde{P}_1 &= P_1\cos(\frac{\varrho}{2}P_0) - P_2\sin(\frac{\varrho}{2}P_0),\\
\widetilde{P}_2 &= P_2\cos(\frac{\varrho}{2}P_0) + P_1\sin(\frac{\varrho}{2}P_0),\\
\widetilde{P}_3 &= P_3,\\
\widetilde{R}_1 &= R_1\cos(\frac{\varrho}{2}P_0) - R_2\sin(\frac{\varrho}{2}P_0),\\
\widetilde{R}_2 &= R_2\cos(\frac{\varrho}{2}P_0) + R_1\sin(\frac{\varrho}{2}P_0),\\
\widetilde{R}_3 &= R_3,\\
\widetilde{N}_1 &= N_1\cos(\frac{\varrho}{2}P_0) - N_2\sin(\frac{\varrho}{2}P_0) + \frac{\varrho}{2}R_3\widetilde{P}_1,\\
\widetilde{N}_2 &= N_2\cos(\frac{\varrho}{2}P_0) + N_1\sin(\frac{\varrho}{2}P_0) + \frac{\varrho}{2}R_3\widetilde{P}_2,\\
\widetilde{N}_3 &= N_3 + \frac{\varrho}{2}R_3P_3.
\end{aligned}
$$

Structure and properties of $U_\varrho(\mathfrak{p})$

The algebra sector remains trivial, while the coproducts and antipodes change.

We find another quantum algebra with the following bicrossproduct structure

$$\mathcal{U}_\varrho(\mathfrak{p}) = U(\mathfrak{so}(1,3)) \blacktriangleright\!\!\lhd \mathcal{T}_\varrho,$$

It is possible to show that this structure is the analogue of the classical semidirect product of groups $G = H \rtimes F$ given by the split short exact sequence

$$H \hookrightarrow G \to F.$$

One has, in fact, that [7]

$$\mathcal{T}_\varrho \hookrightarrow \mathcal{U}_\varrho(\mathfrak{p}) \to U(\mathfrak{so}(1,3)).$$

7. G. Fabiano, G. Gubitosi, F. Lizzi, LS, P. Vitale, *Bicrossproduct vs. twist quantum symmetries in noncommutative geometries : the case of ϱ-Minkowski*, J. High Energ. Phys. 2023, 220 (2023).

Twist framework advantages

The twist operator provides (when the r-matrix satisfies a CYBE) a straightforward way to deform Hopf algebras.

It also naturally induces a noncommutative product ($\star$-product) on coordinates :

$$(f \star g)(x) := \mu_0 \circ \mathcal{F}^{-1}(f \otimes g)(x).$$

It is possible, in this way, to build field theories on noncommutative spacetimes employing $\star$-products.

We will, now, analyze some properties of these $\star$-products and their relation with quantum group structures.

Plane waves and $\star$-products

The $\star$-products allow to define naturally the composition rules of noncommutative plane waves :

$$e^{ip\cdot x} \star e^{ik\cdot x} = e^{i(p\oplus k)\cdot x},$$

in this way we can construct products between generic functions as Fourier expansions

$$(f \star g)(x) = \int d^4p\, d^4k\, \tilde{f}(p)\tilde{g}(k)\; e^{ip_\mu x^\mu} \star e^{ik_\nu x^\nu}\,.$$

The deformed sum $\oplus$ is naturally related to the coproducts :

$$P_\mu \left(\phi(p) \star \phi(k)\right) = P_\mu \phi(p \oplus k) = \mu_0 \circ \Delta P_\mu(\phi(p) \otimes \phi(k)).$$

Plane waves and ⋆-products

The ⋆-product depends on the ordering procedure for the noncommutative exponentials.

The time-to-the-tight ordering

$$\hat{\phi}_R(p) = e^{ip_k \hat{x}^k} e^{ip_0 \hat{x}^0},$$

gives the bicrossproduct coproducts.

The time-symmetric ordering

$$\hat{\phi}_S(p) = e^{i\frac{p_0 \hat{x}^0}{2}} e^{ip_k \hat{x}^k} e^{i\frac{p_0 \hat{x}^0}{2}},$$

gives the twist coproducts.

Different ordering prescriptions correspond to different quantum algebra bases.

Properties of the $\star$-products

Time-to-the right : this product is cyclic with respect to the standard integration measure on $\mathbb{R}^4$:

$$\int d^4x\ f(x) \star g(x) = \int d^4x\ g(x) \star f(x)\ ,$$

but it is not closed :

$$\int d^4x\ f \star g \neq \int d^4x\ f \cdot g\ .$$

Time-symmetric[8] : cyclic and closed.

Cyclicity $\longrightarrow$ allows $\star$-invariant gauge actions.

Closure $\longrightarrow$ undeformed quadratic terms (e.g., mass, kinetic...) $\longrightarrow$ undeformed tree level propagator.

8. M. Dimitrijevic Ciric, N. Konjik, M. A. Kurkov, F. Lizzi, P. Vitale, *Noncommutative field theory from angular twist*, Phys. Rev. D 98 (2018) 085011.

Conclusions : comparing the models

Conclusions

1. We have analyzed the quantum group and the quantum algebra of ϱ-Poincaré,
2. we have proved that the quantum group has a natural bicrossproduct structure,
3. we have found a suitable change of generators for the quantum algebra to be of the bicrossproduct type,
4. we have investigated the differences between the two **complementary** approaches, highliting the adavantages of using one of them instead of the other,
5. we constructed a new kind of $\star$-product in ϱ-Minkowski, compared it with the known one coming from the twist basis and related them to the two studied quantum algerbas of ϱ-Poincaré.

Thanks for your attention !

Bibliography :

1 G. Fabiano, G. Gubitosi, F. Lizzi, LS, P. Vitale, *Bicrossproduct vs. twist quantum symmetries in noncommutative geometries : the case of ϱ-Minkowski*, J. High Energ. Phys. 2023, 220 (2023),

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