Prof. Dr. Dieter Lüst
Theresienstr. 37, Room 425

Falk Haßler
F.Hassler@lmu.de

4. Polyakov action and conformal transformations

To be discussed on Thursday, November 14, 2013 in the tutorial.

Exercise 4.1: Polyakov action (field equations)

Consider the Polyakov action

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}$$

a) Remembering $\det(\exp A) = \exp(\mathrm{Tr}\, A)$, show that

$$\delta h = -h_{\alpha\beta}(\delta h^{\alpha\beta})h\,,$$

where $h = -\det(h_{\alpha\beta})$.

b) The energy momentum tensor $T_{\alpha\beta}$ describes the response of the action to changes in the metric:

$$\delta S = -T\int d^2\sigma\sqrt{h}T_{\alpha\beta}\delta h^{\alpha\beta} \quad\Leftrightarrow\quad T_{\alpha\beta} = -\frac{1}{T\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}}\,.$$

Compute $T_{\alpha\beta}$ for the Polyakov action.

c) Find the equations of motion for $h^{\alpha\beta}$ and show that, after some manipulation and re-insertion into S_{P}, one re-obtains the Nambu-Goto action.

d) Show that adding a "cosmological constant term",

$$S_1 = \lambda\int d^2\sigma\sqrt{h}\,,$$

to the Polyakov action leads to inconsistent field equations for $h_{\alpha\beta}$ in the combined system $S_{\mathrm{P}} + S_1$ when $\lambda \neq 0$.

Exercise 4.2: Polyakov action (symmetries)

a) Show in one line that the Weyl invariance $S_{\mathrm{P}}[e^{2\Lambda}h_{\alpha\beta}, X^\mu] = S_{\mathrm{P}}[h_{\alpha\beta}, X^\mu]$ automatically implies $h^{\alpha\beta}T_{\alpha\beta} = 0$ without the use of the equations of motion.

b) Verify the tracelessness of $T_{\alpha\beta}$ directly by using your result for $T_{\alpha\beta}$ from problem 1b).

c) How does $h_{\alpha\beta}$ have to transform under arbitrary reparameterizations $(\tau,\sigma) \to (\tilde{\tau}(\tau,\sigma), \tilde{\sigma}(\tau,\sigma))$ for S_{P} to be invariant?

Exercise 4.3: The residual conformal transformations

a) Using light cone coordinates $\sigma^{\pm}$, the world sheet metric in conformal gauge reads

$$ds^2 = -\Omega^2 d\sigma^+ d\sigma^- \,,$$

where the conformal factor $\Omega(\sigma^+, \sigma^-)$ can be absorbed by a Weyl transformation to make the metric flat. Show that transformations of the type

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+)\,, \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$$

do not lead one out of the conformal gauge. These transformations are called *conformal transformations* and correspond to a residual freedom in choosing the worldsheet coordinates even after one has gone to conformal gauge.

b) Using $T_{\pm\pm} = \frac{1}{2}\partial_{\pm}X \cdot \partial_{\pm}X$ and the Poisson brackets in conformal gauge,

$$\begin{aligned} \{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} &= \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = 0 \\ \{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} &= \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')\,, \end{aligned}$$

calculate the Poisson brackets

$$\{T_{\pm\pm}(\sigma,\tau), X^\mu(\sigma',\tau)\}\,.$$

c) Use the definition

$$L_\xi := 2T \int_0^{\bar{\sigma}} d\sigma \xi(\sigma^+) T_{++}(\sigma^+)$$

and the result of part b) to calculate the Poisson bracket

$$\{L_\xi, X^\mu(\sigma,\tau)\}$$

and show that the L_ξ generate infinitesimal conformal transformations via the Poisson bracket.

d) For the closed string, one can also define the analogous quantities for T_{--} and decompose the functions $\xi(\sigma^{\pm})$ into Fourier components $e^{im\sigma^{\pm}}$. The resulting generators L_m and $\bar{L}_m$ then form two copies of the classical Virasoro algebra with respect to the Poisson bracket, i.e.

$$\{L_m, L_n\} = -i(m-n)L_{m+n}\,,$$

and similarly for the $\bar{L}_m$. Verify explicitly that the above commutation relations satisfy the Jacobi identity, i.e. form a Lie algebra.

e) Show that the generators L_0, L_1 and L_{-1} form a Lie subalgebra.

f) Show that the combination $(\bar{L}_0 - L_0) = T\int_0^{2\pi} d\sigma \dot{X} \cdot X'$ generates rigid σ-translations along the closed string.