Tutorial for String Theory I, WiSe2013/14
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6. Virasoro Algebra

To be discussed on Thursday, November 28, 2013 in the tutorial.

Exercise 6.1: Normal ordering and the quantum Virasoro algebra

In this exercise, we will show that in the quantized bosonic string theory, the normal ordered Virasoro generators

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\alpha_n:$$

satisfy the Virasoro algebra with a central charge[1]:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n}\,.$$

In order to become more familiar with the normal ordering prescription, we will do this by brute force methods, i.e., by simply using the definition of the normal ordered generators L_m and then calculating their commutators. We will proceed in several smaller steps.

a) Explain why the normal ordering in L_m only affects L_0 and why the Virasoro generators L_m can be written in the following form:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{0}\alpha_n\alpha_{m-n} + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{m-n}\alpha_n\,. \tag{1}$$

b) Using $[X, Y, Z] = [X, Y]Z + Y[X, Z]$ and $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$ prove that, for all $m, n \in \mathbb{Z}$

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}\,.$$

c) Decompose the sum

$$\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{0} + \sum_{n=1}^{\infty}$$

as we did in (1) to "solve" the normal ordering condition. Use the result of part b) to show that

$$[L_m, L_n] = \frac{1}{2}\sum_{l=-\infty}^{0}\big((m-l)\alpha_l\alpha_{m+n-l} + l\alpha_{n+l}\alpha_{m-l}\big) + \frac{1}{2}\sum_{l=1}^{\infty}\big((m-l)\alpha_{m+n-l}\alpha_l + l\alpha_{m-l}\alpha_{n+l}\big)\,. \tag{2}$$

[1] A central charge, T_0, of a Lie algebra is a generator that commutes with all generators of the Lie algebra, $[T_a, T_0] = 0$, but appears on the right hand side of some commutators, $[T_a, T_b] = cT_0 + \dots$, for some T_a and T_b with c being a constant. In the above Virasoro algebra, the role of T_0 is played by the term proportional to δ_{m+n}, which should be viewed as an additional generator in addition to the L_m.

d) Make the substitution $p = n + l$ in the second and forth term in (2) and verify

$$[L_m, L_n] = \frac{1}{2}\Big(\sum_{l=-\infty}^{0} (m-l)\alpha_l \alpha_{m+n-l} + \sum_{p=-\infty}^{n} (p-n)\alpha_p \alpha_{m+n-p} + \sum_{l=1}^{\infty} (m-l)\alpha_{m+n-l}\alpha_l + \sum_{p=n+1}^{\infty} (p-n)\alpha_{m+n-p}\alpha_p \Big) . \quad (3)$$

e) From now on, we will restrict ourselves to the case $n > 0$, as the other case $n < 0$ and $n = 0$ are completely analogous. Show therefore that for $n > 0$, the expression (3) in d) is equal to

$$[L_m, L_n] = \frac{1}{2}\Big(\sum_{p=-\infty}^{0} (m-n)\alpha_p \alpha_{m+n-p} + \sum_{p=1}^{n} (p-n)\alpha_p \alpha_{m+n-p} + \sum_{p=n+1}^{\infty} (m-n)\alpha_{m+n-p}\alpha_p + \sum_{p=1}^{n} (m-p)\alpha_{m+n-p}\alpha_p \Big) . \quad (4)$$

Which of these for terms are already normal-ordered?

f) Prove

$$\sum_{p=1}^{n} (p-n)\alpha_p \alpha_{m+n-p} = \sum_{p=1}^{n} (p-n)\alpha_{m+n-p}\alpha_p + \sum_{p=1}^{n} (p-n) p D \delta_{m+n}$$

and insert this for the second term in the expression (4) of part e).

g) Show that your result from part e) is now equivalent to

$$[L_m, L_n] = \frac{1}{2} \sum_{l=-\infty}^{\infty} (m-n) : \alpha_l \alpha_{m+n-l} : + \frac{1}{2} D \sum_{l=1}^{n} (l^2 - nl)\delta_{m+n} .$$

h) Prove, e.g. by induction, the following identities:

$$\sum_{q=1}^{n} q^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{q=1}^{n} q = \frac{1}{2} n(n+1)$$

and use them to finally derive

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n}$$

from the expression in part g).