Tutorial for String Theory I, WiSe2013/14
Prof. Dr. Dieter Lüst — Falk Haßler
Theresienstr. 37, Room 425 — F.Hassler@lmu.de

3. The 2-sphere

To be discussed on Thursday, November 7, 2013 in the tutorial.

Exercise 3.1: Nambu-Goto action of the 2-sphere

A two-sphere of fixed radius ρ in three-dimensional Euclidean space, $\mathbb{R}^3$, can be considered a Euclidean analogue of an (admittedly some-what peculiar) string world sheet. Using $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$ as the analogue of the world sheet coordinates (τ, σ), the standard spherical coordinates yield the embedding functions

$$\begin{aligned} X^1(\theta, \phi) &= \rho \sin\theta \cos\phi \\ X^2(\theta, \phi) &= \rho \sin\theta \sin\phi \\ X^3(\theta, \phi) &= \rho \cos\theta \end{aligned}$$

which are the analogues of $X^\mu(\tau, \sigma)$ for the usual string.

a) Calculate the matrix

$$M = \begin{pmatrix} \frac{\partial \vec{X}}{\partial \theta} \cdot \frac{\partial \vec{X}}{\partial \theta} & \frac{\partial \vec{X}}{\partial \theta} \cdot \frac{\partial \vec{X}}{\partial \phi} \\ \frac{\partial \vec{X}}{\partial \phi} \cdot \frac{\partial \vec{X}}{\partial \theta} & \frac{\partial \vec{X}}{\partial \phi} \cdot \frac{\partial \vec{X}}{\partial \phi} \end{pmatrix} .$$

b) Calculate the area of the two-sphere using the Euclidean analogue of the Nambu-Goto action:

$$A = \int_0^\pi \int_0^{2\pi} d\theta d\phi \sqrt{\det(M)} .$$

Exercise 3.2: Differential geometry of a 2-sphere

Consider the metric of a 2-sphere of radius a:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2 \left[d\theta^2 + \sin^2\theta d\phi^2 \right] .$$

The metric encodes all information on the geometry of the manifold. We will determine all geometric quantities that are relevant for general relativity:

a) *The metric:* Choosing $x^1 = \theta$ and $x^2 = \phi$, read of the matrix $g_{\mu\nu}$.

b) *The Christoffel symbols:* The Christoffel symbols are defined as

$$\Gamma^\kappa_{\lambda\mu} = \frac{1}{2} g^{\kappa\nu} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right) .$$

They enter covariant derivatives such as $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$, where the correction term with the Christoffel symbols ensures that the covariant derivative indeed transforms "covariantly" under arbitrary coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, i.e.,

$$\nabla_\mu V^\nu \to (\nabla_\mu V^\nu)' = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\rho} \nabla_\lambda V^\rho ,$$

without second derivatives in the coordinates.
Compute the non-vanishing Christoffel symbols for the two-sphere. (Hint: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$, so only a few components have to be computed explicitly.

c) *The Riemann tensor:* The Riemann curvature tensor has the form

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\eta}_{\nu\lambda}\Gamma^{\kappa}_{\mu\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\kappa}_{\nu\eta}\,.$$

Calculate the non-vanishing components of $R^{\kappa}_{\lambda\mu\nu}$ for the two-sphere (Hint: Use the anti-symmetry in μ and ν to avoid redundant computations).

d) *The Ricci tensor:* The Ricci tensor is defined as

$$\mathrm{Ric}_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}\,.$$

Calculate $\mathrm{Ric}_{\mu\nu}$ for S^2.

e) *The scalar curvature:* The scalar curvature is given as

$$\mathcal{R} = g^{\mu\nu}\mathrm{Ric}_{\mu\nu}\,.$$

Calculate $\mathcal{R}$ for S^2. How does the scalar curvature behave in the limit $a \to \infty$? Interpret this behaviour.

f) *The Einstein tensor:* The Einstein equation is the field equation of general relativity. It relates the curvature of spacetime to the matter distribution:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,,$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor:

$$G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\,.$$

Calculate $G_{\mu\nu}$ for S^2.