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5. Classical relativistic string

To be discussed on Thursday, November 21, 2013 in the tutorial.

Exercise 5.1: Oscillator expansion for the closed string

Consider the mode expansion of the closed string:

$$
\begin{aligned}
X^\mu_\mathrm{R} &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)} \\
X^\mu_\mathrm{L} &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\bar\alpha^\mu_n e^{-in(\tau+\sigma)}\,.
\end{aligned}
$$

Use the Poisson brackets

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar\alpha^\mu_m, \bar\alpha^\nu_n\} = -im\delta_{m+n}\eta^{\mu\nu}$$

to reproduce the Poisson brackets

$$
\begin{aligned}
\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} &= \{\dot X^\mu(\sigma,\tau), \dot X^\nu(\sigma',\tau)\} = 0 \\
\{X^\mu(\sigma,\tau), \dot X^\nu(\sigma',\tau)\} &= \frac{1}{T}\eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned}
$$

for X^μ and $\dot X^\mu$ in conformal gauge.

Exercise 5.2: Time evolution of a closed circular string

At $t=0$, a closed string forms a circle of radius R in the x-y-plane and has zero velocity. The time evolution of this string is determined by the action

$$S = -T\int dt \int_0^{\sigma_1} d\sigma \left(\frac{ds}{d\sigma}\right)\sqrt{1-\frac{\vec v_\perp^2}{c^2}}\,,$$

where $\vec v_\perp$ is the component of the velocity $\partial\vec X/\partial t$ in the direction perpendicular to the string. The string will remain circular but its radius will become a *time-dependent* function $R(t)$.

a) Give the Lagrangian L in terms of $R(t)$ and its time derivatives.
b) Calculate the radius and velocity of the string as functions of time.
c) Sketch the spacetime surface traced by the string in a three-dimensional plot using x, y and ct as axes.

Hint: Calculate the Hamiltonian associated with L and use energy conservation.

Exercise 5.3: Hamiltonian density for the relativistic string

Consider the Lagrangian density $\mathcal{L}$ in the static gauge and written in terms of $\partial_\sigma \vec{X}$, $\partial_t \vec{X}$. Show that the canonical momentum density $\vec{\mathcal{P}}(t,\sigma)$ is given by

$$\vec{\mathcal{P}}(t,\sigma) = \frac{T}{c^2} \frac{\vec{v}_\perp}{\sqrt{1-\frac{\vec{v}_\perp^2}{c^2}}} \frac{ds}{d\sigma} \,.$$

a) Calculate the Hamiltonian density $\mathcal{H}$, again in terms of $\vec{v}_\perp$ and $ds/d\sigma$.

b) Write the total Hamiltonian $H = \int d\sigma \mathcal{H} = \int ds(\ldots)$.

c) Show that your answer is consistent with the interpretation that the *energy* of the string arises as energy of the *transverse* motion of the string whose *rest mass* arises entirely from its *tension*.