Tutorial for String Theory I, WiSe2013/14
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12. String compactification on the torus

To be discussed on Thursday, January 23, 2014 in the tutorial.

Exercise 12.1: Torus compactification

Just as a circle can equivalently be described as $\mathbb{R}/2\pi R\mathbb{Z}$, a d-dimensional Torus, T^d, can be described as $\mathbb{R}^d/2\pi\Lambda^d$, where Λ^d denotes a d-dimensional lattice generated by integer linear combinations of d basis vectors $(i,j=1,\dots,d)$,

$$\vec{V}_i = \frac{1}{\sqrt{2}} R_i \vec{e}_i \qquad \text{(no sum)}.$$

Here, the vectors $\vec{e}_i$ are normalized as

$$\vec{e}_i \cdot \vec{e}_i = 2\,,$$

so that $\vec{V}_i$ has length R_i. Denoting the Cartesian components of the vectors $\vec{e}_i$ by e_i^I $(I,J = 1,\dots,d)$, the torus with radii R_i is then given by the identification

$$X^I \sim X^I + 2\pi \sum_{i=1}^{d} V_i^I n_i \equiv X^I + 2\pi L^I\,, \quad (n_i \in \mathbb{Z}) \tag{1}$$

where

$$L^I := \sum_{i=1}^{d} V_i^I n_i = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} n_i R_i e_i^I \tag{2}$$

are the Cartesian coordinates of the possible lattice vectors. A set of basis vectors $\vec{e}_i^{\,*}$ is said to be dual to the basis $\vec{e}_i$ if

$$\vec{e}_i \cdot \vec{e}_j^{\,*} \equiv \sum_{I=1}^{d} e_i^I e_j^{*I} = \delta_{ij}$$

which also implies

$$\sum_{i=1}^{d} e_i^I e_i^{*J} = \delta^{IJ}\,.$$

The Euclidean metric δ_{IJ} on $\mathbb{R}^d$ can be expressed in terms of the bases $\vec{V}_i$ or $V_i^* = \frac{\sqrt{2}}{R_i}\vec{e}_i^{\,*}$, in terms of which it reads

$$g_{ij} = \vec{V}_i \cdot \vec{V}_j = \frac{1}{2}\sum_{I=1}^{d} R_i e_i^I R_j e_j^I$$

$$g_{ij}^* = \vec{V}_i^* \cdot \vec{V}_j^* = 2\sum_{I=1}^{d} \frac{1}{R_i} e_i^{*I} \frac{1}{R_j} e_j^{*I}\,.$$

g_{ij} and g_{ij}^* play the rôle of the metric on, respectively, the lattice Λ^d and the dual lattice $(\Lambda^d)^*$ generated by $\vec{V}_i^*$.

a) Show that g_{ij}^* is actually the inverse of g_{ij}.

b) Choosing $\vec{e}_1 = (\sqrt{2}, 0)$ and $\vec{e}_2 = (1, 1)$ in case of a 2-torus, find the dual basis $\vec{e}_i^*$ (either graphically or algebraically).

c) Show that the single-valuedness of a wave function of the form $\exp\left(\sum_{I=1}^{d} X^I p^I\right)$ requires

$$\sum_{I=1}^{d} L^I p^I \in \mathbb{Z}\,,$$

and hence,

$$p^I = \sum_{i=1}^{d} m_i V_i^{*I}\,, \quad (m_i \in \mathbb{Z})\,.$$

The momentum vectors p^I are thus constrained to lie on the dual lattice $(\Lambda_d)^*$.

d) The mass formula in the absence of internal B_{MN} fields is given by

$$m^2 = N_L + N_R - 2 + \frac{1}{2}\left((\vec{p}_L)^2 + (\vec{p}_R)^2\right)\,,$$

where

$$\vec{p}_{L,R} = \vec{p} \pm \frac{1}{2}\vec{L}\,.$$

Show that this is equal to

$$m^2 = N_L + N_R - 2 + \sum_{i,j=1}^{d}\left(m_i g_{ij}^* m_j + \frac{1}{4} n_i g_{ij} n_j\right)\,.$$

e) Switching now on a non-trivial internal B_{MN}-field background, $B_{IJ} \neq 0$, and using a flat spacetime metric, $G_{MN} = \eta_{MN}$, the action of a string (string tension $T = \frac{1}{4}\pi$)

$$\begin{aligned} S \equiv S_{\mathrm{P}} + S_{\mathrm{B}} = &-\frac{1}{8\pi}\int d^2\sigma\left(-\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\right)\eta_{MN} \\ &+ \frac{1}{4\pi}\int d^2\sigma \partial_\tau X^I \partial_\sigma X^J B_{IJ}\,. \end{aligned}$$

Use

$$X^I(\sigma,\tau) = x^I + 2p^I + L^I\sigma + \text{oscillators}\,,$$

to show that the internal canonical momenta

$$\Pi^I = \frac{\delta S}{\delta(\partial_\tau X^I)}$$

are given by

$$\Pi^I = \frac{1}{2\pi}(p^I + \frac{1}{2}B_{IJ}L^J) \;+\; \text{oscillators}\,.$$

This implies that the internal canonical center of mass momenta π^I are now given by

$$\pi^I = p^I + \frac{1}{2}B_{IJ}L^J$$

instead of just p^J. Hence, we now have to require single-valuedness of $\exp\left(i\sum_l \pi^I X^I\right)$ instead of $\exp\left(i\sum_l p^I X^I\right)$, so that π^I and not p^I is now quantized:

$$\pi^I = \sum_{i=1}^{d} m_i V_i^{*I} .$$

f) While the canonical momentum has changed, it is still the mechanical momentum p^I that enters $p^I_{L,R}$, just as in (2), and the mass formula is still of the form (1). Reexpressing p^I in terms of π^I, show that

$$\begin{aligned} p^I_{L,R} &= \pi^I \pm \frac{1}{2}(\delta^{IJ} \mp B_{IJ})L^J \\ &= (m_i - \frac{1}{2}n_j B_{ij})V_i^{*I} \pm \frac{1}{2}n_i V_i^I , \end{aligned} \tag{3}$$

where sums over repeated indices are understood and $B_{ij} \equiv \sum_{I,J} V_i^I V_j^J B_{IJ}$.

Remark: Inserting (3) into (1), one finds that the mass again depends on g_{ij} (and its dual/inverse g^*_{ij}), but also on B_{ij}. Hence, there are $d(d+1)/2 + d(d-1)/2 = d^2$ continuous parameters g_{ij}, B_{ij} that label the different physically inequivalent configurations. In the low energy effective field theory, these parameters ("moduli") arise as the vev's of d^2 lower-dimensional scalar fields, which are simply zero modes of the internal metric and 2-form field components. The scalar potential of these scalar fields is (classically) flat, and so their vev's are not dynamically fixed. Finding mechanisms that fix the moduli of string compactifications is an important problem in present day string theory research, and much progress has been achieved in recent years in this area.