Tutorial for String Theory I, WiSe2013/14
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7. Quantization of the relativistic string

To be discussed on Thursday, December 5, 2013 in the tutorial.

Exercise 7.1: Old covariant quantization

In the old covariant quantization procedure, all fields $X^\mu(\tau,\sigma)$ $\big(\mu = 0,\dots,(D-1)\big)$ are kept as dynamical variables. The corresponding naive Fock space $\mathcal{H}_{\text{Fock}}$, which is generated by action with all possible combinations of raising operators α^μ_{-m} (and $\bar\alpha^\mu_{-m}$ for the closed string) $(m > 0)$ on the Fock vacuum $|0,p\rangle$, always contains negative norm states. These negative norm states, however, are completely harmless provided they are all projected out by the physical state conditions

$$L_m\,|\text{phys}\rangle = 0 \quad (m>0) \tag{1}$$
$$(L_0 - 1)\,|\text{phys}\rangle = 0 \tag{2}$$

and similarly for the $\bar L_m$ in the case of the closed string. These physical state conditions are the quantum implementation of the classical Virasoro constraints.

a) Use the second of these constraint, eq. (2), as well as the relation $\sqrt{2\alpha'}p^\mu = \alpha^\mu_0$ to derive the mass shell condition for the open string:

$$\alpha' m^2 = (N-1)\,, \tag{3}$$

where

$$N = \sum_{m>0} \alpha_{-m}\alpha_m$$

denotes the number operator. States that satisfy the relation (3) automatically solve the constraint (2).

b) Consider the following open string state:

$$|\Phi\rangle = \frac{1}{2}\left[\alpha_{-1}\alpha_{-1} + \frac{(D-1)}{5}p\alpha_{-2} + \frac{(D+4)}{10}(p\alpha_{-1})^2\right]|0,p\rangle\ .$$

In the rest of this problem, we set

$$\alpha' = \frac{1}{2}\,.$$

We now want to verify whether this state is physical, i.e., whether this state satisfies (1) and (2). Use first (3) from part a) to derive the constraint imposed on $p^\mu p_\mu = -m^2$ by (2).

c) Use $[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}$ and (remembering $\alpha' = \frac{1}{2}$) $p^\mu = \alpha^\mu_0$ to show that $|\Phi\rangle$ is annihilated by L_1, L_2 and $L_{m>3}$. This implies that $|\Phi\rangle$ is physical for all spacetime dimensions D.

d) Calculate the norm of $|\Phi\rangle$ and show that there are negative norm states in the physical spectrum for $D > 26$.

Exercise 7.2: Path integral and Faddeev-Popov determinant

The Polyakov action $S_{\mathrm{P}}[X^\mu, h_{\alpha\beta}]$ treats the embedding coordinates X^μ and the world sheet metric $h_{\alpha\beta}$ as dynamical variables. S_{P} is invariant under the (infinite-dimensional) group *diff*× *Weyl* of world sheet diffeomorphisms (corresponding to arbitrary reparameterizations $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$) and local Weyl rescalings of the metric $h_{\alpha\beta} \to e^{2\Lambda(\tau,\sigma)} h_{\alpha\beta}$. These gauge symmetries allow one to classically eliminate $h_{\alpha\beta}$ as a dynamical variable by going to the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$.
In the *quantum* theory, this gauge fixing requires some more care and might even be impossible, as is best seen in the path integral formulation (For a more detailed account on path integrals and the Faddeev-Popov procedure see e.g. Peskin Schroeder, Ch. 9 and 16.) The naive vacuum amplitude, or partition function,

$$Z = \int \frac{\mathcal{D}[h]\mathcal{D}[X]}{V_{\mathit{diff}\times \mathit{Weyl}}} e^{iS_{\mathrm{P}}} \tag{4}$$

sums over all possible field configurations $[X^\mu(\tau,\sigma), h_{\alpha\beta}]$ between some fixed initial and final values and weighs them with the exponential of the classical action. This path integral contains a huge overcounting, as all gauge equivalent field configurations are independently integrated over. Formally, one should therefore normalize this expression by dividing by the "volume" $V_{\mathit{diff}\times \mathit{Weyl}}$ of the local symmetry group, which, however, is itself infinite. In order to make the naive expression (4) more meaningful, one should therefore use a change of integration variables

$$\mathcal{D}[h]\mathcal{D}[X] \to \mathcal{D}[\text{gauge equivalent}]\mathcal{D}[\text{gauge inequivalent}]$$

so that redundant integration over gauge equivalent configurations can be factored out and formally be "canceled" by the volume factor, leaving an integration over the physically independent configurations only. Just as for finite-dimensional integrals this change of variables comes with a Jacobian, the so-called Faddeev-Popov determinant Δ_{FP}, which has to be included in the remaining integration over the gauge inequivalent configurations,

$$Z = \int \mathcal{D}[\text{gauge inequivalent}]\Delta_{\mathrm{FP}} e^{iS_{\mathrm{P}}} \, .$$

For the factorization into integrations over gauge equivalent and gauge inequivalent configurations to be possible, the integration measure of the original path integral has to be gauge invariant. If this is not the case, the gauge degrees of freedom cannot be consistently decoupled in the quantum theory, and the theory becomes anomalous.
In our case, the gauge symmetries act non-trivially on $h_{\alpha\beta}$, their infinitesimal action being given by (c.f. the lecture)

$$\begin{aligned} \delta h_{\alpha\beta} &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + 2\Lambda h_{\alpha\beta} \\ &= (P\xi)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta} \end{aligned}$$

with $(P\xi)_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi^\gamma) h_{\alpha\beta}$ and $2\tilde{\Lambda} = 2\Lambda + (\nabla_\gamma \xi^\gamma)$. As $h_{\alpha\beta}$ is completely gauged, one can write

$$\mathcal{D}[h] = \mathcal{D}[P\xi]\mathcal{D}[\tilde{\Lambda}] = \mathcal{D}[\xi]\mathcal{D}[\Lambda] \left| \frac{\delta(P\xi, \tilde{\Lambda})}{\delta(\xi, \Lambda)} \right| \, .$$

a) Using $\frac{\delta(P\xi)}{\delta\xi} = P$, show that, formally, the matrix

$$\begin{pmatrix} \frac{\delta(P\xi)}{\delta\xi} & \frac{\delta(P\xi)}{\delta\Lambda} \\ \frac{\delta\tilde{\Lambda}}{\delta\xi} & \frac{\tilde{\Lambda}}{\delta\Lambda} \end{pmatrix}$$

has lower triangular form.

b) Use this to infer that, formally,

$$\left| \frac{\delta(P\xi, \tilde{\Lambda})}{\delta(\xi, \Lambda)} \right| = \det P \,.$$

Hence, $\det P$ play the role of the Faddeev-Popov determinant, and one has

$$Z = \int \mathcal{D}[X](\det P) e^{iS_{\mathrm{P}}[X, h_{\alpha\beta} = \eta_{\alpha\beta}]} \,.$$

Exercise 7.3: Analytic confinuation of the zeta function

Consider the definition of the gamma function and the Riemann zeta function,

$$\Gamma(s) = \int_0^\infty dt e^{-t} t^{s-1} \,, \quad \zeta(s) = \sum_{n=1}^\infty n^{-s} \,.$$

a) Prove that

$$\Gamma(s)\zeta(s) = \int_0^\infty dt \frac{t^{s-1}}{e^t - 1} \,, \Re(s) > 1 \,.$$

b) Verify the small t expansion

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2) \,.$$

c) Use the above equation to show that for $\Re(s) > 1$

$$\Gamma(s)\zeta(s) = \int_0^1 dt t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{s-1} + \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty dt \frac{t^{s-1}}{e^t - 1} \,.$$

d) Explain why the right-hand side above is well defined also for $\Re(s) > -2$. It then follow that this right-hand side defines an analytic continuation of the left-hand side to $\Re(s) > -2$.

e) Recall the pole structure of $\Gamma(s)$ and use it to show that

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12} \,.$$

Argue that the zeta function regularization one would therefore conclude that

$$\sum_{n=1}^\infty = -\frac{1}{12} \,.$$