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Quantum Field Theory – Practice Exam

21st of June 2022

Please fill in:

Name: ______________________

Matriculation Number: ______________________

Number of Sheets: ______________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each individual problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.

Do not write below this line.

Comments:

Question:	1	2	Total
Points:	19	25	44
Score:			

1. Scalar QED

19 points

Consider the Largrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - \frac{\lambda}{4} |\phi|^4 \,,$$

where ϕ is a complex scalar field.

(a) (2 points) Find the equations of motion for ϕ.

(b) (1 point) The theory has a one continuous U(1) symmetry. Find it.

(c) (2 points) Derive the Noether current associated to the symmetry you found.

(d) (2 points) Verify that the current you found is conserved if the equations of motion are satisfied.

(e) (4 points) Gauge the U(1) symmetry of the theory. Use the minimal coupling to couple the scalar field to a electromagnetic field. Write down the resulting Lagrangian (including the kinetic term for the gauge field). Write down the local symmetry transformation under which this Lagrangian is invariant.

(f) (3 points) Find the classical equations of motion for all fields in this theory (including the gauge field).

(g) (2 points) Compare the Noether current obtained from the original Lagrangian (before gauging it) to the current appearing on the RHS of Maxwell's equations that you found now. Which of them is gauge invariant?

(h) (3 points) Sketch the Feynman diagrams describing interactions of the theory. Which interaction term is new and isn't there in QED with fermionic matter?

2. And its renormalisation 25 points

The Lagrangian for scalar electrodynamics has the form

$$\mathcal{L} = D_\mu \phi (D^\mu \phi)^* - m^2 \phi \phi^* - \frac{\lambda}{4} |\phi|^4 + F^{\mu\nu} F_{\mu\nu}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \frac{1}{4}\partial_\nu A_\mu$.

(a) (3 points) What are the counter-terms needed to renormalise the theory?

(b) (2 points) Draw the Feynman graphs generating the one-loop corrections to the scalar propagator.

(c) (4 points) Show that in $d = 4 - \epsilon$ dimensions, in the $\lambda = 0$ limit, the one loop corrections to the propagator have the form

$$i\Pi_\phi(k^2) = 4e^2\mu^\epsilon \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{P_{\alpha\beta}(l) k^\alpha k^\beta}{l^2\left((l+k)^2 + m^2\right)} - 2(d-1)e^2\mu^\epsilon \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{l^2 + m_\gamma^2} \, .$$

Here μ is a constant with scaling dimension 1, $P_\mu(l) = g_{\mu\nu} - \frac{l_\mu l_\nu}{l^2}$ and m_γ is an infra-red regulator.

(d) (2 points) Show that the second term on the right hand side (RHS) vanishes in the limit $m_\gamma \to 0$ (which ϵ fixed).

(e) (7 points) Show that the first term on the RHS can be written as

$$4e^2\mu^\epsilon \int_0^1 \mathrm{d}x \, \frac{x}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^2k^2 - (q\cdot k)^2}{(q^2 + X)^3}$$

where $X = x(1-x)k^2 + xm^2$.

(f) (7 points) To $\mathcal{O}(\epsilon^{-1})$ determine the one-loop counter terms needed to renormalise $\Pi_\phi(k^2)$.

Hints:

1. *Feynman rules in Lorentz gauge:*
   - *scalar:* $-\frac{i}{k^2+m^2}$
   - *photon:* $-\frac{iP_{\mu\nu}(k)}{k^2}$
   - *scalar-scalar-photon vertex,* $ie(k+k')_\mu$, *k and k' are incoming and outgoing scalar 4-momenta*
   - *scalar-scalar-photon-photon vertex,* $-2ie^2 g_{\mu\nu}$.
2. *Feynman parameter formula:*

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA + (1-x)B)^2} \, .$$

3. *In* $d = 4 - \epsilon$ *dimensions*

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2a}}{(k^2+\Delta)^b} = i\frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a+d/2)\Gamma(b-a-d/2)}{\Gamma(b)\Gamma(d/2)}$$

where $\Gamma(\epsilon) = \epsilon^{-1} + \mathcal{O}(\epsilon)$.