Poisson-Lie T-duality in Double Field Theory

Falk Hassler

University of North Carolina at Chapel Hill
University of Pennsylvania

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and

1502.02428 with Pascal du Bosque, Dieter Lüst and Ralph Blumenhagen

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Motivation

- Abelian T-Duality
- Double Field Theory
- Generalized Geometry
- Doubled Geometry

Chris Hull
Barton Zwiebach
Olaf Hohm
Motivation

T-Duality

abelian

Double Field Theory

Generalized Geometry

Doubled Geometry

T-Duality

non-abelian

Poisson-Lie

Chris Hull
Barton Zwiebach
Olaf Hohm
Fernando Quevedo
Yolanda Lozano
Ctirad Klimcik
Daniel Thompson

Doubled

Geometry

Double Field Theory

Generalized Geometry

time
Motivation

abelian T-Duality

non-abelian
Poisson-Lie

Doubled Geometry

Double Field Theory

Generalized Geometry

abelian T-Duality

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Daniel Thompson
Dieter Lüst
Ralph Blumenhagen
Daniel Waldram
Charles S-C. . .
Outline

1. Motivation
2. Poisson-Lie T-duality
3. Double Field Theory on Drinfeld doubles
4. Application: Dilaton transformation
5. Summary
**Drinfeld double** [Drinfeld, 1988]

**Definition:** A **Drinfeld double** is a 2D-dimensional Lie group $\mathcal{D}$, whose Lie-algebra $\mathfrak{d}$

1. has an $\text{ad}$-invariant bilinear form $\langle \cdot, \cdot \rangle$ with signature $(D, D)$
2. admits the decomposition into two maximal isotropic subalgebras $\mathfrak{g}$ and $\mathfrak{g}$
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2. admits the decomposition into two maximal isotropic subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$

$$(t^a \ t_a) = t_A \in \mathfrak{d}, \quad t_a \in \mathfrak{g} \quad \text{and} \quad t^a \in \tilde{\mathfrak{g}}$$

$$\langle t_A, t_B \rangle = \eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^b_a & 0 \end{pmatrix}$$
Drinfeld double [Drinfeld, 1988]

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\[
\begin{aligned}
(t^a \ t_a) &= t_A \in \mathfrak{d}, \quad t_a \in \mathfrak{g} \quad \text{and} \quad t^a \in \tilde{\mathfrak{g}} \\
\langle t_A, t_B \rangle &= \eta_{AB} = 
\begin{pmatrix}
0 & \delta^a \ b \\
\delta^b \ a & 0
\end{pmatrix}
\\
[t_A, t_B] &= F_{AB}^C t_C \quad \text{with non-vanishing commutators}
\end{aligned}
\]

\[
\begin{aligned}
[t_a, t_b] &= f_{ab}^c t_c \\
[t_a, t^b] &= \tilde{f}^{bc} a t_c - f_{ac}^b t^c \\
[t^a, t^b] &= \tilde{f}^{ab} c t^c
\end{aligned}
\]

$\text{ad}$-invariance of $\langle \cdot , \cdot \rangle$ implies $F_{ABC} = F_{[ABC]}$
Poisson-Lie T-duality: 1. Definition [Klimcik and Severa, 1995]

- 2D $\sigma$-model on target space $M$ with action
  \[ S(E, M) = \int dzd\bar{z} \ E_{ij} \partial x^i \bar{\partial} x^j \]
- $E_{ij} = g_{ij} + B_{ij}$ captures metric and two-form field on $M$
- inverse of $E_{ij}$ is denoted as $E^{ij}$
Poisson-Lie T-duality: 1. Definition [Klimcik and Severa, 1995]

- 2D $\sigma$-model on target space $M$ with action
  \[ S(E, M) = \int dz d\bar{z} E_{ij} \partial x^i \partial \bar{x}^j \]
- $E_{ij} = g_{ij} + B_{ij}$ captures metric and two-from field on $M$
- inverse of $E_{ij}$ is denoted as $E^{ij}$
- *left* invariant vector field $\nu_a^i$ on $G$ is the inverse transposed of *right* invariant Maurer-Cartan form $t_a \nu^a_i dx^i = dg g^{-1}$
- adjoint action of $g \in G$ on $t_A \in \mathfrak{g}$: $\text{Ad}_g t_A = g t_A g^{-1} = M_A^B t_B$
- analog for $\tilde{G}$
Poisson-Lie T-duality: 1. Definition [Klimcik and Severa, 1995]

- 2D $\sigma$-model on target space $M$ with action
  \[ S(E, M) = \int dzd\tilde{z} E_{ij} \partial x^i \partial x^j \]
- $E_{ij} = g_{ij} + B_{ij}$ captures metric and two-from field on $M$
- inverse of $E_{ij}$ is denoted as $E^{ij}$
- left invariant vector field $v^a_i$ on $G$ is the inverse transposed of right invariant Maurer-Cartan form $t_a v^a_i dx^i = dg g^{-1}$
- adjoint action of $g \in G$ on $t_A \in \mathfrak{g}$: $\text{Ad}_g t_A = g t_A g^{-1} = M_A^B t_B$
- analog for $\tilde{G}$

**Definition:** $S(E, \mathcal{D}/\tilde{G})$ and $S(\tilde{E}, \mathcal{D}/G)$ are Poisson-Lie T-dual if

\[
E^{ij} = v^i_c M^c_a (M^{ae} M^b_e + E_0^{ab}) M^d_b v^j_d \\
\tilde{E}^{ij} = \tilde{v}^c_i \tilde{M}^a_c (\tilde{M}^{ae} \tilde{M}^b_e + E_0^{ab}) \tilde{M}^d_b \tilde{v}^{d_j}
\]

holds, where $E_0^{ab}$ is constant and invertible with the inverse $E_0^{-1}$. 

Motivation

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<th>Application</th>
<th>Summary</th>
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</table>
Poisson-Lie T-duality: 2. Properties

- captures
  - abelian T-d. $G$ abelian and $\tilde{G}$ abelian
  - non-abelian T-d. $G$ non-abelian and $\tilde{G}$ abelian

[Ossa and Quevedo, 1993; Giveon and Rocek, 1994; Alvarez, Alvarez-Gaume, and Lozano, 1994; ...]
Poisson-Lie T-duality: 2. Properties

- captures \[ \begin{cases} \text{abelian T-d.} & G \text{ abelian and } \tilde{G} \text{ abelian} \\ \text{non-abelian T-d.} & G \text{ non-abelian and } \tilde{G} \text{ abelian} \end{cases} \]

  \[ \text{[Ossa and Quevedo, 1993; Giveon and Rocek, 1994; Alvarez, Alvarez-Gaume, and Lozano, 1994; ...]} \]

- dual \( \sigma \)-models related by canonical transformation

  \[ \text{[Klimcik and Severa, 1995; Klimcik and Severa, 1996; Sfetsos, 1998]} \]

- equivalent at the classical level

- preserves conformal invariance at one-loop

  \[ \text{[Alekseev, Klimcik, and Tseytlin, 1996; Sfetsos, 1998; ...; Jurco and Vysoky, 2017]} \]
Poisson-Lie T-duality: 2. Properties

- captures
  \[\begin{aligned}
  &\text{abelian T-d.} & G \text{ abelian} & \text{and } \tilde{G} \text{ abelian} \\
  &\text{non-abelian T-d.} & G \text{ non-abelian} & \text{and } \tilde{G} \text{ abelian}
  \end{aligned}\]
  
  [Ossa and Quevedo, 1993; Giveon and Rocek, 1994; Alvarez, Alvarez-Gaume, and Lozano, 1994; ...]

- dual \(\sigma\)-models related by canonical transformation
  
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- equivalent at the classical level

- preserves conformal invariance at one-loop
  
  [Alekseev, Klimcik, and Tseytlin, 1996; Sfetsos, 1998; ...; Jurco and Vysoky, 2017]

- dilaton transformation [Jurco and Vysoky, 2017]

\[\begin{aligned}
\phi &= -\frac{1}{2} \log \left| \det \left( 1 + \tilde{g}_0^{-1}(\tilde{B}_0 + \Pi) \right) \right| \\
\tilde{\phi} &= -\frac{1}{2} \log \left| \det \left( 1 + g_0^{-1}(B_0 + \tilde{\Pi}) \right) \right| \\
\phi &= \text{details later}
\end{aligned}\]
Additional structure on the Drinfeld double


- right invariant vector $E_A^I$ field on $\mathcal{D}$ is the inverse transposed of left invariant Maurer-Cartan form $t_A E^A_I dX^I = g^{-1} dg$
Additional structure on the Drinfeld double


- right invariant vector $E_A^I$ field on $\mathcal{D}$ is the inverse transposed of left invariant Maurer-Cartan form $t_AE^A_I dX^I = g^{-1} dg$

- two $\eta$-compatible, covariant derivatives\(^1\)
  1. flat derivative
     \[
     D_A V^B = E_A^I \partial_I V^B - wF_A V^B, \quad F_A = D_A \log |\det(E^B_I)|
     \]
  2. convenient derivative
     \[
     \nabla_A V^B = D_A V^B + \frac{1}{3} F_{AC}^B V^C
     \]

\(^1\)definitions here just for quantities with flat indices
Additional structure on the Drinfeld double

- *Right* invariant vector $E_A^I$ field on $\mathcal{D}$ is the inverse transposed of *left* invariant Maurer-Cartan form $t_A E^A_I dX^I = g^{-1} dg$

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     $$\nabla_A V^B = D_A V^B + \frac{1}{3} F_{AC}^B V^C$$

- generalized metric $\mathcal{H}_{AB} (w = 0)$
  $$\mathcal{H}_{AB} = \mathcal{H}_{(AB)}, \quad \mathcal{H}_{AC} \eta^{CD} H_{DB} = \eta_{AB}$$

- generalized dilatton $d$ with $e^{-2d}$ scalar density of weight $w = 1$

---

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  \[ \mathcal{H}_{AB} = \mathcal{H}_{(AB)}, \quad \mathcal{H}_{AC} \eta^{CD} H_{DB} = \eta_{AB} \]
- generalized dilaton $d$ with $e^{-2d}$ scalar density of weight $w = 1$
- triple $(\mathcal{D}, \mathcal{H}_{AB}, d)$ captures the doubled space of DFT

$^1$definitions here just for quantities with flat indices
Double Field Theory for \((\mathcal{D}, \mathcal{H}_{AB}, d)\) [Blumenhagen, Bosque, Hassler, and Lüst, 2015]

see also [Vaisman, 2012; Hull and Reid-Edwards, 2009; Geissbuhler, Marques, Nunez, and Penas, 2013; Cederwall, 2014; ...]

- action \((\nabla_A d = -\frac{1}{2} e^{2d} \nabla_A e^{-2d})\)

\[
S_{NS} = \int_{\mathcal{D}} d^{2D} X e^{-2d} \left( \frac{1}{8} \mathcal{H}^{CD} \nabla_C \mathcal{H}_{AB} \nabla_D \mathcal{H}^{AB} - \frac{1}{2} \mathcal{H}^{AB} \nabla_B \mathcal{H}^{CD} \nabla_D \mathcal{H}_{AC} - 2 \nabla_A d \nabla_B \mathcal{H}^{AB} + 4 \mathcal{H}^{AB} \nabla_A d \nabla_B d + \frac{1}{6} F_{ACD} F_B^{CD} \mathcal{H}^{AB} \right)
\]
Double Field Theory for $\mathcal{D}, \mathcal{H}_{AB}, d$ [Blumenhagen, Bosque, Hassler, and Lüst, 2015]
see also [Vaisman, 2012; Hull and Reid-Edwards, 2009; Geissbuhler, Marques, Nunez, and Penas, 2013; Cederwall, 2014; ...]

- action ($\nabla_A d = -\frac{1}{2} e^{2d} \nabla_A e^{-2d}$)

$$S_{NS} = \int_D d^{2D} X e^{-2d} \left( \frac{1}{8} \mathcal{H}^{CD} \nabla_C \mathcal{H}_{AB} \nabla_D \mathcal{H}^{AB} - \frac{1}{2} \mathcal{H}^{AB} \nabla_B \mathcal{H}^{CD} \nabla_D \mathcal{H}_{AC} ight)$$

- 2D-diffeomorphisms

$$L_\xi V^A = \xi^B D_B V^A + w D_B \xi^B V^A$$

- global $O(D,D)$ transformations

$$V^A \rightarrow T^A_B V^B \quad \text{with} \quad T^A_C T^B_D \eta^{CD} = \eta^{AB}$$
Double Field Theory for $(D, \mathcal{H}_{AB}, d)$ [Blumenhagen, Bosque, Hassler, and Lüst, 2015]

see also [Vaisman, 2012; Hull and Reid-Edwards, 2009; Geissbuhler, Marques, Nunez, and Penas, 2013; Cederwall, 2014; ... ]

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- 2 \nabla_A d \nabla_B \mathcal{H}^{AB} + 4 \mathcal{H}^{AB} \nabla_A d \nabla_B d + \frac{1}{6} F_{ACD} F_B^{CD} \mathcal{H}^{AB} \right)$$

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- global $O(D,D)$ transformations

$$V^A \rightarrow T^A_B V^B \quad \text{with} \quad T^A_C T^B_D \eta^{CD} = \eta^{AB}$$

- generalized diffeomorphisms

$$\mathcal{L}_\xi V^A = \xi^B \nabla_B V^A + (\nabla^A \xi_B - \nabla_B \xi^A) V^B + w \nabla_B \xi^B V^A$$

- section condition (SC)

$$\eta^{AB} D_A \cdot D_B \cdot = 0$$
Symmetries of the action

- $S_{NS}$ invariant for $X^I \rightarrow X^I + \xi^A E_A^I$ and

1. $\mathcal{H}^{AB} \rightarrow \mathcal{H}^{AB} + L_{\xi} \mathcal{H}^{AB}$ and $e^{-2d} \rightarrow e^{-2d} + L_{\xi} e^{-2d}$
2. $\mathcal{H}^{AB} \rightarrow \mathcal{H}^{AB} + L_{\xi} \mathcal{H}^{AB}$ and $e^{-2d} \rightarrow e^{-2d} + L_{\xi} e^{-2d}$
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<table>
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<tr>
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<th>gen.-diffeomorphisms</th>
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<th>global $O(D,D)$</th>
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<tr>
<td>$\mathcal{H}^{AB}$</td>
<td>tensor</td>
<td>scalar</td>
<td>tensor</td>
</tr>
<tr>
<td>$\nabla_A d$</td>
<td>not covariant</td>
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<td>$e^{-2d}$</td>
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<td>invariant</td>
</tr>
<tr>
<td>$\eta^{AB}$</td>
<td>invariant</td>
<td>invariant</td>
<td>invariant</td>
</tr>
<tr>
<td>$F^{ABC}$</td>
<td>invariant</td>
<td>invariant</td>
<td>tensor</td>
</tr>
<tr>
<td>$E_A^I$</td>
<td>invariant</td>
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<tr>
<td>$S_{NS}$</td>
<td>invariant</td>
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<td>invariant</td>
</tr>
<tr>
<td>$SC$</td>
<td>invariant</td>
<td>invariant</td>
<td>invariant</td>
</tr>
<tr>
<td>$D_A$</td>
<td>not covariant</td>
<td>covariant</td>
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manifest
Poisson-Lie T-duality: 1. Solve SC [Hassler, 2016]

- fix $D$ physical coordinates $x^i$ from $X^I = \begin{pmatrix} x^i & \tilde{x}^i \end{pmatrix}$ on $\mathcal{D}$ such that $\eta^{IJ} = E_A^I \eta^{AB} E_B^J = \begin{pmatrix} 0 & \cdots \\ \cdots & \cdots \end{pmatrix} \rightarrow$ SC is solved

- fields and gauge parameter depend just on $x^i$
Poisson-Lie T-duality: 1. Solve SC [Hassler, 2016]

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- fields and gauge parameter depend just on $x^i$

- only two SC solutions, relate them by symmetries of DFT

$$d(X^I) = g(x^i)\tilde{g}(x^{\tilde{i}}) \quad \quad t_A = (t^a \ t_a)$$
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\[
d(X^I) = g(x^i) \tilde{g}(x^\tilde{i}) \quad t_A = (t^a, t_a)
\]

\[
d(X'^I) = \tilde{g}(x'^i) g(x'^\tilde{i}) \quad t^A = (t_a, t^a)
\]
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$$d(X^I) = g(x^i) \tilde{g}(x^i) \quad t_A = (t^a, t_a)$$

$$d(X''^I) = \tilde{g}(x'^i) g(x'^i) \quad t^A = (t_a, t^a)$$

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Poisson-Lie T-duality
Double Field Theory
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Poisson-Lie T-duality: 2. As manifest symmetry of DFT

- same structure as in the original paper [Klimcik and Severa, 1995]
- duality target spaces arise as different solutions of the SC
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Poisson-Lie T-duality:

- 2D-diffeomorphisms $X^I \rightarrow X'^I(X^1, \ldots X^{2D})$ with $d(X^I) = d(X'^I)$
- global $O(D,D)$ transformation $t_A \rightarrow \eta^{AB} t_B$

manifest symmetries of DFT
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manifest symmetries of DFT

- for abelian T-duality $X^I \rightarrow X''^I = X^I$
  $\rightarrow$ no 2D-diffeomorphisms needed, only global $O(D,D)$ transformation
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manifest symmetries of DFT

- for abelian T-duality $X^I \rightarrow X''^I = X^I$
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Poisson-Lie T-duality is a manifest symmetry of DFT
Equivalence to supergravity: 1. Generalized parallelizable spaces

[Lee, Strickland-Constable, and Waldram, 2014]

- generalized tangent space element $V^I = (V^i \ V_i)$
- generalized Lie derivative

$$\mathcal{L}_\xi V^I = \xi^J \partial_J V^I + (\partial^I \xi^J - \partial_J \xi^I) V^J$$

with

$$\partial^i = (0 \ \partial_i)$$
Equivalence to supergravity: 1. Generalized parallelizable spaces

[Lee, Strickland-Constable, and Waldram, 2014]

- generalized tangent space element \( \hat{V}^I = (V^i \ V_i) \)
- generalized Lie derivative

\[
\hat{L}_\xi \hat{V}^I = \xi^J \partial_J \hat{V}^I + (\partial^I \xi_J - \partial_J \xi^I) \hat{V}^J \quad \text{with} \quad \partial_i = (0 \ \partial_i)
\]

Definition: A manifold \( M \) which admits a globally defined generalized frame field \( \hat{E}_A^I(x^i) \) satisfying

1. \( \hat{L}_{\hat{E}_A} \hat{E}_B^I = F_{AB}^C \hat{E}_C^I \)

where \( F_{AB}^C \) are the structure constants of a Lie algebra \( \mathfrak{h} \)

2. \( \hat{E}_A^I \eta^{AB} \hat{E}_B^J = \eta^{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \)

is a generalized parallelizable space \( (M, \mathfrak{h}, \hat{E}_A^I) \).
Drinfeld double $\mathcal{D} \rightarrow$ two generalized parallelizable spaces:

$$(D/\tilde{G}, \vartheta, \hat{E}_A^\hat{l})$$

and

$$(D/G, \vartheta, \tilde{E}_A^\hat{l})$$

\[
\hat{E}_A^\hat{l} = M_A^B \begin{pmatrix} v^b_i & 0 \\ 0 & \nu^b_i \end{pmatrix} B^\hat{l}
\]

\[
\tilde{E}_A^\hat{l} = \tilde{M}_{AB} \begin{pmatrix} \tilde{v}_b^i & 0 \\ 0 & \tilde{v}^b_i \end{pmatrix} B^\hat{l}
\]
Drinfeld double \( \mathcal{D} \to \) two generalized parallelizable spaces:

\[
\begin{aligned}
\left( \mathcal{D} / \tilde{\mathcal{G}}, \varphi, \tilde{E}_A \right) \\
\left( \mathcal{D} / \mathcal{G}, \varphi, \tilde{E}_A \right)
\end{aligned}
\]

and

\[
\begin{aligned}
\tilde{E}_A = M_A^B \left( \begin{array}{cc}
v^b_i & 0 \\
0 & v_b^i
\end{array} \right) B \tilde{\mathcal{B}}
\end{aligned}
\]

express \( \mathcal{H}^{AB} \) in terms of the generalized \( \hat{\mathcal{H}}^{ij} \) on \( \mathcal{T} \mathcal{D} / \tilde{\mathcal{G}} \oplus \mathcal{T}^{\ast} \mathcal{D} / \tilde{\mathcal{G}} \)

\[
\mathcal{H}^{AB} = \hat{E}_A \hat{\mathcal{H}}^{ij} \hat{E}_B \hat{\mathcal{J}}
\]

with

\[
\hat{\mathcal{H}}^{ij} = \left( \begin{array}{ccc}
g_{ij} & -B_{ik} g^{kl} B_{lk} & -B_{ik} g^{kl} \\
B_{ik} g^{kl} B_{lk} & g_{ik} & B_{kj} \\
g_{ij} & B_{kj} & g_{ij}
\end{array} \right)
\]

express \( d \) in terms of the standard generalized dilaton \( \hat{d} \)

\[
\begin{aligned}
d &= \hat{d} - \frac{1}{2} \log |\det \tilde{v}_{ai}| \\
\hat{d} &= \phi - \frac{1}{4} \log |\det g_{ij}|
\end{aligned}
\]
Equivalence to supergravity: 2. Generalized metric and dilaton

[Klimcik and Severa, 1995; Hull and Reid-Edwards, 2009; du Bosque, Hassler, Lüst, 2017]

- Drinfeld double $\mathcal{D} \rightarrow$ two generalized parallelizable spaces:

\[
\left( \mathcal{D}/\tilde{G}, \circ, \tilde{E}_A^\hat{I} \right) \quad \text{and} \quad \left( \mathcal{D}/G, \circ, \tilde{E}_A^\hat{I} \right)
\]

\[
\tilde{E}_A^\hat{I} = M_A^B \left( \begin{array}{cc} v_{bi} & 0 \\ 0 & v_{b^i} \end{array} \right) B^\hat{I}
\]

- express $\mathcal{H}^{AB}$ in terms of the generalized $\tilde{\mathcal{H}}^{\hat{I}\hat{J}}$ on $TD/\tilde{G} \oplus T^*D/\tilde{G}$

\[
\mathcal{H}^{AB} = \tilde{E}_A^\hat{I} \tilde{\mathcal{H}}^{\hat{I}\hat{J}} \tilde{E}_B^\hat{J}
\]

with

\[
\tilde{\mathcal{H}}^{\hat{I}\hat{J}} = \left( \begin{array}{ccc} g_{ij} - B_{ik} g^{kl} B_{lk} & -B_{ik} g^{kl} \\ g^{ik} B_{kj} & g^{ij} \end{array} \right)
\]

- express $d$ in terms of the standard generalized dilaton $\hat{d}$

\[
d = \hat{d} - \frac{1}{2} \log |\det \tilde{v}_{ai}|
\]

\[
\hat{d} = \phi - 1/4 \log |\det g_{ij}|
\]

- plug into the DFT action $S_{NS}$

Motivation
Poisson-Lie T-duality
Double Field Theory
Application
Summary
Equivalence to supergravity: 3. IIA/B bosonic sector action

- If $G$ and $\tilde{G}$ are unimodular

$$S_{NS} = V_{\tilde{G}} \int d^D x \ e^{-2\hat{d}} \left( \frac{1}{8} \hat{H}^{KL} \partial_{\hat{K}} \hat{H}^{\hat{I}\hat{J}} \partial_{\hat{L}} \hat{H}^{\hat{I}\hat{J}} - 2 \partial_{\hat{I}} \hat{d} \partial_{\hat{J}} \hat{H}^{\hat{I}\hat{J}} \right. $$

$$\left. - \frac{1}{2} \hat{H}^{\hat{I}\hat{J}} \partial_{\hat{J}} \hat{H}^{\hat{K}\hat{L}} \partial_{\hat{L}} \hat{H}^{\hat{I}\hat{K}} + 4 \hat{H}^{\hat{I}\hat{J}} \partial_{\hat{I}} \hat{d} \partial_{\hat{J}} \hat{d} \right)$$

- $V_{\tilde{G}} = \int_{\tilde{G}} d\tilde{x}^D \ det \tilde{\nu}_{ai}$ volume of group $\tilde{G}$
Equivalence to supergravity: 3. IIA/B bosonic sector action

- If $G$ and $\tilde{G}$ are unimodular

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- $V_{\tilde{G}} = \int_{\tilde{G}} d\tilde{x}^D \ \det \tilde{v}_{ai} \ \text{volume of group} \ \tilde{G}$

- Equivalent to IIA/B NS/NS sector action
  
  [Hohm, Hull, and Zwiebach, 2010; Hohm, Hull, and Zwiebach, 2010]

$$S_{NS} = V_{\tilde{G}} \int d^D x \ \sqrt{\det(g_{ij})} e^{-2\phi} \left( \mathcal{R} + 4\partial_i \phi \partial^i \phi - \frac{1}{12} H_{ijk} H^{ijk} \right)$$

- Holds for all $\mathcal{H}_{AB}(x^i) / \hat{\mathcal{H}}^{\hat{I}\hat{J}}(x^i)$

- Only $D$-diffeomorphisms and $B$-field gauge trans. as symmetries
Equivalence to supergravity: 3. IIA/B bosonic sector action

- if $G$ and $\tilde{G}$ are unimodular
  \[
  S_{\text{NS}} = V_{\tilde{G}} \int d^D x \, e^{-2 \tilde{d}} \left( \frac{1}{8} \hat{H}^{KL} \partial_K \hat{H}^{IJ} \partial_L \hat{H}^{IJ} - 2 \partial_i \tilde{d} \partial_j \hat{H}^{IJ} \right. \\
  \left. - \frac{1}{2} \hat{H}^{IJ} \partial_j \hat{H}^{KL} \partial_L \hat{H}^{IK} + 4 \hat{H}^{IJ} \partial_i \tilde{d} \partial_j \tilde{d} \right)
  \]

- $V_{\tilde{G}} = \int_{\tilde{G}} d\tilde{x}^D \det \tilde{\nu}_{ai}$ volume of group $\tilde{G}$

- equivalent to IIA/B NS/NS sector action
  [Hohm, Hull, and Zwiebach, 2010; Hohm, Hull, and Zwiebach, 2010]
  \[
  S_{\text{NS}} = V_{\tilde{G}} \int d^D x \sqrt{\det(g_{ij})} e^{-2\phi} (R + 4 \partial_i \phi \partial^i \phi - \frac{1}{12} H_{ijk} H^{ijk})
  \]

- holds for all $\mathcal{H}_{AB}(x^i) / \hat{H}^{IJ}(x^i)$

- only $D$-diffeomorphisms and $B$-field gauge trans. as symmetries

- similar story for R/R sector
Restrictions on $\mathcal{H}_{AB}$ and $d$ to admit Poisson-Lie T-duality

- in general $\mathcal{H}_{AB}(x^i) \xrightarrow{\text{Poisson-Lie T-duality (2D-diff.)}} \mathcal{H}_{AB}(x'^i, x'^\tilde{i})$
- $x'^\tilde{i}$ part not compatible with ansatz for SUGRA reduction $\rightarrow$ avoid it
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A doubled space $(D, \mathcal{H}_{AB}, d)$ admits Poisson-Lie T-dual supergravity descriptions iff

1. $L_\xi \mathcal{H}_{AB} = 0 \quad \forall \xi \quad \rightarrow \quad D_A \mathcal{H}_{AB} = 0$
2. $L_\xi d = 0 \quad \forall \xi \quad \rightarrow \quad D_A e^{-2d} = 0$
Application: Dilaton transformation

\[ D_A e^{-2d} = 0 \quad \rightarrow \quad \partial_I \left( 2d + \log |\det v| + \log |\det \tilde{v}| \right) = 0 \]
\[ = 2\phi_0 = \text{const.} \]
Application: Dilatation transformation

1. \( D_A e^{-2d} = 0 \) \(\rightarrow\) \( \partial_I (2d + \log |\det v| + \log |\det \tilde{v}|) = 0 \)
   \(\Rightarrow\) \( 2\phi_0 = \text{const.} \)

2. \( d = \phi - 1/4 \log |\det g| - \frac{1}{2} \log |\det \tilde{v}| \)
   \(\phi = \phi_0 + \frac{1}{4} \log |\det g| - \frac{1}{2} \log |\det v| \)
Application: Dilaton transformation

$D_A e^{-2d} = 0 \quad \rightarrow \quad \partial_I (2d + \log |\det v| + \log |\det \tilde{v}|) = 0$

$= 2\phi_0 = \text{const.}$

$d = \phi - 1/4 \log |\det g| - \frac{1}{2} \log |\det \tilde{v}|$

$\phi = \phi_0 + \frac{1}{4} \log |\det g| - \frac{1}{2} \log |\det v|$

$g = v^T e^T e v \quad \text{with} \quad \begin{cases} (\tilde{B}_0 + \tilde{g}_0)^{ab} = E^{0\,ab} \\ \Pi^{ab} = M^{ac} M^{b\,c} \\ e^{-1} e^{-T} = \tilde{g}_0 - (\tilde{B}_0 + \Pi) \tilde{g}_0^{-1} (\tilde{B}_0 + \Pi) \\ \tilde{e}_0^T \tilde{e}_0 = \tilde{g}_0 \\ e^{-T} = \tilde{e}_0 + \tilde{e}_0^{-T} (\tilde{B}_0 + \Pi) \end{cases}$
Application: Dilaton transformation

- \( D_A e^{-2d} = 0 \quad \rightarrow \quad \partial_I (2d + \log |\det v| + \log |\det \tilde{v}|) = 0 \)

\( = 2\phi_0 = \text{const.} \)

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\( \phi = \phi_0 + \frac{1}{4} \log |\det g| - \frac{1}{2} \log |\det v| \)

\( (\tilde{B}_0 + \tilde{g}_0)^{ab} = E^{0 \, ab} \)

\( \Pi^{ab} = M^{ac} M^{b \, c} \)

\( e^{-1} e^{-T} = \tilde{g}_0 - (\tilde{B}_0 + \Pi)\tilde{g}_0^{-1} (\tilde{B}_0 + \Pi) \)

\( \tilde{e}_0^T \tilde{e}_0 = \tilde{g}_0 \)

\( e^{-T} = \tilde{e}_0 + \tilde{e}_0^{-T} (\tilde{B}_0 + \Pi) \)

- \( \phi = \phi_0 + \frac{1}{2} \log |\det e| = \phi_0 - \frac{1}{2} \log |\det \tilde{e}_0| - \frac{1}{2} \log \left| \det \left( 1 + \tilde{g}_0^{-1} (\tilde{B}_0 + \Pi) \right) \right| \)

- reproduces [Jurco and Vysoky, 2017]
Summary

- DFT, Poisson-Lie T-duality and Drinfeld doubles fit together naturally
- interpretation of doubled space does not require winding modes anymore (phase space perspective instead)

Various new directions for research in DFT

- Translation of all the intriguing results in Poisson-Lie T-duality
  [Klimcik and Severa, 1996; Sfetsos, 1998; Klimcik, and Severa, 1996 (momentum ↔ winding); ...]

- Drinfeld doubles → quantum groups → rich mathematical structure

- New way to organized $\alpha'$ corrections?
- New way to construct non-geometric backgrounds?
- Branes in curved space [Klimcik, and Severa, 1996 (D-branes)]
- Facilitates new applications

- Integrable deformations of 2D $\sigma$-models (see Daniel's talk)

- Solution generating technique

- Explore underlying structure of AdS/CFT (see Yolandia's talk)
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Hull and Zwiebach, 2009

Klimcik, 2002
Big picture

- Poisson-Lie
- T-duality
- Double Field Theory
- Generalized parallelizable space
- Drinfeld double
- Worldsheet
- Target space

Double Field Theory

Generalized parallelizable space